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A note on a broken Dirichlet convolution

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Abstract: The paper deals with a broken Dirichlet convolution $\otimes$ which is based on using the odd divisors of integers. In addition to presenting characterizations of $\otimes$-multiplicative functions we also show an analogue of Menon's identity:

$$\sum_{\substack{a \ (mod \ n) \\ (a,n)_\otimes = 1}} (a-1, n) = \phi_\otimes(n)[\tau(n) - \frac{1}{2}\tau_2(n)],$$

where $(a, n)_\otimes$ denotes the greatest common odd divisor of a and n, $\phi_\otimes(n)$ is the number of integers a (mod n) such that $(a, n)_\otimes = 1$, $\tau(n)$ is the number of divisors of n, and $\tau_2(n)$ is the number of even divisors of n.

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1 Introduction

An arithmetical function is a complex-valued function whose domain is the set of positive integers $\mathbb{Z}^+$. The Dirichlet convolution $f * g$ of two arithmetical function f and g is defined by

$$(f * g)(n) = \sum_{d|n} f(d) g\Big(\frac{n}{d}\Big),$$

where the summation is over all the divisors d of n (the term "divisor" always means "positive divisor"). The identity element relative to the Dirichlet convolution is the function δ:

$$\delta(n) = \begin{cases} 1 & \text{if} \quad n = 1 \\ 0 & \text{otherwise} \end{cases}$$

An arithmetical function f has a convolution inverse if and only if $f(1) \neq 0$. The convolution inverse of the zeta function ζ ($\zeta(n) = 1$ for any $n \in \mathbb{Z}^+$) is the (classical) Möbius function μ:

$$\mu(n) = \begin{cases} 1 & \text{if} \quad n = 1 \\ (-1)^k & \text{if} \quad n \text{ is a product of } k \text{ distinct primes} \\ 0 & \text{if} \quad n \text{ has one or more repeated prime factors.} \end{cases}$$

There are many fundamental results about algebras of arithmetical functions with a variety of convolutions. The Davison$-$ or $K-$convolution ([2], [10, Chapter 4]) $f *_K g$ of two arithmetical functions f and g is defined by

$$(f *_K g)(n) = \sum_{d|n} K(n,d) f(d) g\Big(\frac{n}{d}\Big),$$

where K is a complex-valued function on the set of all pairs of positive integers (n, d) with $d|n$. If $K \equiv 1$ then the K-convolution is the Dirichlet convolution.

In [12] the $\mathbb{C}$-algebra of extended arithmetical functions is considered as an incidence algebra of a proper Möbius category. If a category C is decomposition-finite (i.e. C is a small category in which for any morphism α, $\alpha \in MorC$, there are only a finite number of pairs $(\beta, \gamma) \in MorC \times MorC$ such that $\gamma\beta = \alpha$) then the C-convolution $\widetilde{f} * \widetilde{g}$ of two incidence functions $\widetilde{f}$ and $\widetilde{g}$ (that is two complex-valued functions defined on the set $MorC$ of all morphisms of C) is defined by:

$$(\widetilde{f} * \widetilde{g})(\alpha) = \sum_{\gamma\beta=\alpha} \widetilde{f}(\beta)\widetilde{g}(\gamma).$$

The incidence function $\widetilde{\delta}$ defined by

$$\widetilde{\delta}(\alpha) = \begin{cases} 1 & \text{if } \alpha \text{ is an identity morphism} \\ 0 & \text{otherwise} \end{cases}$$

is the identity element relative to the C-convolution $*$. A Möbius category (in the sense of Leroux [9, 1]) is a decomposition-finite category in which an incidence function $\widetilde{f}$ has a convolution inverse if and only if $\widetilde{f}(\alpha) \neq 0$ for any identity morphism α. The Möbius function $\widetilde{\mu}$ of a Möbius category C is the convolution inverse of the zeta function $\widetilde{\zeta}$ defined by $\widetilde{\zeta}(\alpha) = 1$ for any morphism α of C. Some useful characterizations of a Möbius category C are given in [1, 7, 8, 9]. The set of all incidence functions $I(C)$ of a Möbius category C becomes a $\mathbb{C}$-algebra with the usual pointwise addition and multiplication and the C-convolution $*$.

The prime example of a Möbius category (with a single object) is the multiplicative monoid of positive integers $\mathbb{Z}^+$, the convolution being the Dirichlet convolution and the associated Möbius function being the classical Möbius function. A simple example of a proper Möbius category is the category $C_\otimes$ with two objects 1 and 2 and with $Hom_{C_\otimes}(1,1) = 2\mathbb{Z}^+ - 1$ (the set of odd positive integers), $Hom_{C_\otimes}(1,2) = 2\mathbb{Z}^+$ (the set of even positive integers), $Hom_{C_\otimes}(2,1) = \emptyset$, $Hom_{C_\otimes}(2,2) = \{id_2\}$, the composition of morphisms being the usual multiplication of integers. In this case, the $C_\otimes$-convolution (called the broken Dirichlet convolution in [12]) $\widetilde{f} \otimes \widetilde{g}$ of two incidence functions $\widetilde{f}$ and $\widetilde{g}$ is the following one:

$$n \in \mathbb{Z}^+, \quad (\widetilde{f} \otimes \widetilde{g})(n) = \widetilde{f}(n)\widetilde{g}(id_2) + \sum_{\substack{vu=n;\ u\neq n \\ u\in 2\mathbb{Z}^+-1}} \widetilde{f}(u)\widetilde{g}(v); \quad (\widetilde{f} \otimes \widetilde{g})(id_2) = \widetilde{f}(id_2)\widetilde{g}(id_2).$$

In [12] the elements of the incidence algebra $I(C_\otimes)$ are called extended arithmetical functions. Now,

$$\mathcal{A} = \{\widetilde{f} \in I(C_\otimes) | \widetilde{f}(id_2) = \widetilde{f}(1)\}$$

is a subalgebra of the incidence algebra $I(C_\otimes)$ (see [12, Remark 4.2]). All elements of this subalgebra $\mathcal{A}$ are arithmetical functions and the convolution induced in $\mathcal{A}$ for arithmetical functions is the following:

$$n \in \mathbb{Z}^+, \quad (f \otimes g)(n) = f(n)g(1) + \sum_{\substack{d|n;\ d<n \\ d\in 2\mathbb{Z}^+-1}} f(d)g\Big(\frac{n}{d}\Big).$$

It is straightforward to see that the above arithmetical functions convolution is a Davison convolution with:

$$K_\otimes(n,d) = \begin{cases} 1 & \text{if } d = n \text{ or } d \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that the incidence functions $\widetilde{\delta}, \widetilde{\zeta}, \widetilde{\mu} \in I(C_\otimes)$ are elements of the subalgebra $\mathcal{A}$ and, as arithmetical functions, they coincide with the arithmetical functions δ, ζ and $\mu_\otimes$ respectively, where (see [12, Proposition 2.1])

$$\mu_\otimes(n) = \begin{cases} \mu(n) & \text{if } \ n \text{ is odd} \\ -1 & \text{if } \ n = 2^k \ (k > 0) \\ 0 & \text{if } \ n \text{ is even, } n \neq 2^k. \end{cases}$$

2 Odd-multiplicative arithmetical functions

Following Haukkanen [3], an arithmetical function f is K-multiplicative (where K is the basic complex-valued function of a Davison convolution) if

(1) $f(1) = 1$;

(2) $(\forall n \in \mathbb{Z}^+)$, $f(n)K(n,d) = f(d)f(\frac{n}{d})K(n,d)$, for all $d|n$.

In the case of a Möbius category C we say that an incidence function $f \in I(C)$ is C-multiplicative (see also [11]) if the following conditions hold:

(1) $f(1) = 1$;

(2) $(\forall \alpha \in MorC)$, $f(\alpha) = f(\beta)f(\gamma)$, for all $(\beta,\gamma) \in MorC \times MorC$ with $\gamma\beta = \alpha$.

Now, we call an arithmetical function f odd-multiplicative if

(1) $f(1) = 1$;

(2) $(\forall n \in \mathbb{Z}^+)$, $f(n) = f(2^{n(2)}) \prod_p [f(p)]^{n(p)}$, where $n = 2^{n(2)} \prod_p p^{n(p)}$ is the canonical factorization of n.

Proposition 2.1. *Let f be an arithmetical function. The following statements are equivalent:*

(i) *f is odd-multiplicative;*

(ii) *f is $C_{\otimes}$-multiplicative;*

(iii) *f is $K_{\otimes}$-multiplicative.*

Proof. $(i) \Rightarrow (ii)$. Let $n = 2^{n(2)} \prod_p p^{n(p)}$ be the canonical factorization of n and let $n = vu$ the product of two positive integers u and v such that u is odd. If $u = 2^{u(2)} \prod_p p^{u(p)}$ and $v = 2^{v(2)} \prod_p p^{v(p)}$ are the canonical factorizations of u and v respectively then $u(2) = 0$, $u(p) \leq n(p)$ and $v(2) = n(2)$, $v(p) = n(p) - u(p)$. It follows:

$$f(n) = f(2^{n(2)}) \prod_p [f(p)]^{n(p)} = f(2^{u(2)}) \prod_p [f(p)]^{u(p)} f(2^{v(2)}) \prod_p [f(p)]^{v(p)} = f(u)f(v).$$

$(ii) \Rightarrow (iii)$. Id d is an odd divisor of n then $n = \frac{n}{d} d$ is a factorization of the morphism n in $C_{\otimes}$. Therefore $f(n) = f(d)f(\frac{n}{d})$. Since $K_{\otimes}(n,d) = 0$ if d is even, it follows:

$$f(n)K_{\otimes}(n,d) = f(d)f(\frac{n}{d})K_{\otimes}(n,d) \quad \text{for all } d|n.$$

$(iii) \Rightarrow (i)$. Let $n = 2^{n(2)} \prod_p p^{n(p)}$ be the canonical factorization of n. Since $\prod_p p^{n(p)}$ is an odd divisor of n it follows:

$$f(n) = f(2^{n(2)}) f(\prod_p p^{n(p)}).$$

It remains to be shown that $f(\prod_p p^{n(p)}) = \prod_p [f(p)]^{n(p)}$ which immediately follows by induction. □

Proposition 2.2. *Let f be an arithmetical function such that $f(1) \neq 0$. The following statements are equivalent:*

(i) *f is odd-multiplicative;*

(ii) $f(g \otimes h) = fg \otimes fh$ *for any two arithmetical functions g and h;*

(iii) $f(g \otimes g) = fg \otimes fg$ *for any arithmetical function g;*

(iv) $f\tau_{\otimes} = f \otimes f$, *where*

$$\tau_{\otimes}(n) = \begin{cases} \tau(n) & \text{if } n \text{ is odd} \\ 1 + \tau(m) & \text{if } n = 2^k m,\ k > 0, \text{ and } m \text{ is odd} \end{cases}$$

($\tau(n)$ is the number of divisors of n).

Proof. $(i) \Rightarrow (ii)$.

$$(fg \otimes fh)(n) = f(n)g(n)h(1) + \sum_{\substack{d|n;\ d<n \\ d\in 2\mathbb{Z}^+-1}} f(d)g(d)f(\frac{n}{d})h(\frac{n}{d}) =$$

$$= f(n)[g(n)h(1) + \sum_{\substack{d|n;\ d<n \\ d\in 2\mathbb{Z}^+-1}} g(d)h(\frac{n}{d})] = [f(g \otimes h)](n).$$

$(ii) \Rightarrow (iii)$. This is obvious.

$(iii) \Rightarrow (iv)$. It is straightforward to check that $\zeta \otimes \zeta = \tau_\otimes$. When we put $g = \zeta$ in (iii) we obtain (iv).

$(iv) \Rightarrow (i)$. Since $f(1) = f(1)\tau_\otimes(1) = f(1)f(1)$ and $f(1) \neq 0$, it follows $f(1) = 1$. Now, let $n = 2^{n(2)} \prod_p p^{n(p)}$ be the canonical factorization of n. We shall prove by induction on $s = n(2) + \sum_p n(p)$ that

$$f(n) = f(2^{n(2)}) \prod_p [f(p)]^{n(p)}.$$

If $s = 1$ then obviously the equality holds. The equality holds also if $n = 2^k$. So, we assume that $s > 1$ and in the same time that $\tau_\otimes(n) > 2$. We have

$$f(n)\tau_\otimes(n) = 2f(n) + \sum_{\substack{d|n;\ d\neq 1,n \\ d\in 2\mathbb{Z}^+-1}} f(d)f(\frac{n}{d}).$$

Since $d|n$ and $d \neq 1, n$ it follows, by the hypothesis of induction, that

$$f(d)f(\frac{n}{d}) = f(2^{d(2)}) \prod_p [f(p)]^{d(p)} f(2^{\frac{n}{d}(2)}) \prod_p [f(p)]^{\frac{n}{d}(p)} = f(2^{n(2)}) \prod_p [f(p)]^{n(p)}.$$

Taking into account that $\zeta \otimes \zeta = \tau_\otimes$, we have

$$\sum_{\substack{d|n;\ d\neq 1,n \\ d\in 2\mathbb{Z}^+-1}} f(d)f(\frac{n}{d}) = (\tau_\otimes(n) - 2) f(2^{n(2)}) \prod_p [f(p)]^{n(p)},$$

and therefore

$$f(n) = f(2^{n(2)}) \prod_p [f(p)]^{n(p)}.$$

□

An arithmetical function f is called multiplicative if $f(mn) = f(m)f(n)$ whenever $(m, n) = 1$. If f is multiplicative and $f(1) \neq 0$ (i.e. f is not identically zero) then $f(1) = 1$ and $f^{-1}(1) = 1$. Here and in the next Proposition, f^{-1} (g^{-1}, $(fg)^{-1}$) means the inverse of f (g, fg) relative to the convolution $\otimes$. Note that $C_\otimes$ being a Möbius category, $f(1) \neq 0$ assures the existence of the convolution inverse f^{-1}.

Proposition 2.3. *Let f be a multiplicative arithmetical function such that $f(1) \neq 0$. The following statements are equivalent:*

(i) *f is odd-multiplicative;*

(ii) $fg^{-1} = (fg)^{-1}$ *for any arithmetical function g with* $g(1) \neq 0$*;*

(iii) $f\mu_{\otimes} = f^{-1}$*;*

(iv) $f^{-1}(p^m) = 0$ *for any odd prime* p *and any* $m > 1$.

Proof. $(i) \Rightarrow (ii)$. $\delta = f\delta = f(g \otimes g^{-1}) = fg \otimes fg^{-1}$ and $fg^{-1} \otimes fg = f(g^{-1} \otimes g) = f\delta = \delta$.

$(ii) \Rightarrow (iii)$. $f\mu_{\otimes} = f\zeta^{-1} = (f\zeta)^{-1} = f^{-1}$.

$(iii) \Rightarrow (iv)$. $f^{-1}(p^m) = f(p^m)\mu_{\otimes}(p^m) = f(p^m)\mu(p^m) = 0$ if $m > 1$.

$(iv) \Rightarrow (i)$. Let $n = 2^{n(2)} \prod_p p^{n(p)}$ be the canonical factorization of n. Since f is multiplicative it follows:

$$f(n) = f(2^{n(2)}) \prod_p f(p^{n(p)}).$$

Now, $0 = (f \otimes f^{-1})(p^m) = f(p^m) + f(p^{m-1})f^{-1}(p)$ for any odd prime p and $m \geq 1$. Thus, $f^{-1}(p) = -f(p)$ and $f(p^m) = f(p^{m-1})f(p)$. Therefore,

$$f(n) = f(2^{n(2)}) \prod_p [f(p)]^{n(p)}.$$

□

3 The analogue of Menon's identity

As a matter of course, the Dirichlet convolution leads us to the divisibility relation on $\mathbb{Z}^+$ and the convolution $\otimes$ leads us to an "odd-divisibility" relation $|_{\otimes}$ defined by

$$m|_{\otimes}n \text{ if and only if } m \textit{ is odd and } m|n.$$

We denote the greatest common odd divisor of m and n by $(m, n)_{\otimes}$ and let $\phi_{\otimes}(n)$ be the number of integers a (mod n) such that $(a, n)_{\otimes} = 1$.

Lemma 3.1. *We have:*

(1) $(a, n)_{\otimes} = (a + n, 2n)_{\otimes}$;

(2) $\phi_{\otimes}(2n) = 2\phi_{\otimes}(n)$;

Proof. (1). If $(a, n)_{\otimes} = d$ then d is odd, $d|a$ and $d|n$. It follows that $d|a + n$ and $d|2n$. Therefore, $d|(a + n, 2n)_{\otimes}$. If d' is an odd integer such that $d'|a + n$ and $d'|2n$ then $d'|n$ and $d'|a$. It follows $(a + n, 2n)_{\otimes}|d$, and in conclusion, $(a, n)_{\otimes} = (a + n, 2n)_{\otimes}$.

(2) follows immediately from (1). □

By induction on k, using Lemma 3.1.(2), we obtain the following result.

Proposition 3.1. *Let* $n = 2^k m$ *be the factorization of* n *such that* m *is odd. Then*

$$\phi_{\otimes}(n) = 2^k \phi(m),$$

where ϕ *is Euler's totient function.*

Corollary 3.1. *We have*

$$\phi_{\otimes}(n) = \begin{cases} \phi(n) & \text{if } n \text{ is odd} \\ 2\phi(n) & \text{if } n \text{ is even.} \end{cases}$$

Corollary 3.2. *The arithmetical function $\phi_{\otimes}$ is multiplicative.*

In the theory of arithmetical functions a well known and elegant result is Menon's identity ([6]):

$$\sum_{\substack{a \ (mod \ n) \\ (a,n)=1}} (a-1, n) = \phi(n)\tau(n).$$

In this section, using Menon's generalized identity established by Haukkanen [5], we evaluate the sum

$$\sum_{\substack{a \ (mod \ n) \\ (a,n)_{\otimes}=1}} (a-1, n)$$

which obviously becomes the above expression in the case if n is odd.

In [4], Haukkanen introduced the concept of a generalized divisibility relation (of type $f = \{f_p : p \text{ is prime}\}$) satisfying certain conditions (see also [5, Section 2]). For such a generalized divisibility relation $\wr$, f_p are functions from $\mathbb{Z}^+$ to $\mathbb{Z}^+ \cup \{0\}$ defined by: $f_p(a)$ is the smallest integer $i \in \{1, 2, \cdots a\}$ such that $p^i \wr p^a$ if such i exists, and $f_p(a) = 0$ otherwise. Now, $(m, n)_{\wr}$ denotes the greatest element among the divisors d of m satisfying $d \wr n$ and $\phi_{\wr}(n)$ is the number of integers a (mod n) such that $(a, n)_{\wr} = 1$ (see [5, Section 3]). In [5, Theorem 4.1], Haukkanen established Menon's generalized identity. In particular (see [5, (4.4)],

$$\sum_{\substack{a \ (mod \ n) \\ (a,n)_{\wr}=1}} (a-1, n) = \phi_{\wr}(n) \sum_{d|n} \frac{\phi(d) n_d}{d\phi_{\wr}(n_d)},$$

where $n_d = \prod_{p|d} p^{n(p)}$.

Proposition 3.2. *We have*

$$\sum_{\substack{a \ (mod \ n) \\ (a,n)_{\otimes}=1}} (a-1, n) = \phi_{\otimes}(n)[\tau(n) - \frac{1}{2}\tau_2(n)],$$

where $\tau_2(n)$ is the number of even divisors of n.

Proof. It is straightforward to check that the relation $|_{\otimes}$ is a Haukkanen's generalized divisibility relation of type $f = (0_2, \zeta, \zeta, \cdots)$, where $0_2(a) = 0$ for any positive integer a. Since

$$\sum_{\substack{d|n \\ d \in 2\mathbb{Z}^+ - 1}} \frac{\phi(d) n_d}{d\phi_{\otimes}(n_d)} = \sum_{\substack{d|n \\ d \in 2\mathbb{Z}^+ - 1}} \frac{\phi(d) n_d}{d\phi(n_d)} = \sum_{\substack{d|n \\ d \in 2\mathbb{Z}^+ - 1}} 1 =$$

$$= \text{the number of odd divisors of } n,$$

and

$$\sum_{\substack{d|n\\ d\in 2\mathbb{Z}^+}} \frac{\phi(d)n_d}{d\phi_\otimes(n_d)} \overset{(n_d=2^{n(d)}m_d)}{=} \sum_{\substack{d|n\\ d\in 2\mathbb{Z}^+}} \frac{\phi(d)n_d}{d2^{n_d(2)}\phi(m_d)} =$$

$$= \sum_{\substack{d|n\\ d\in 2\mathbb{Z}^+}} \frac{n_d d\prod_{p|d}\left(1-\frac{1}{p}\right)}{d2^{n_d(2)}m_d\prod_{p|d;p\neq 2}\left(1-\frac{1}{p}\right)} = \sum_{\substack{d|n\\ d\in 2\mathbb{Z}^+}} \left(1-\frac{1}{2}\right) = \frac{1}{2}\tau_2(n),$$

it follows that

$$\sum_{\substack{a \pmod n\\ (a,n)_\otimes=1}} (a-1,n) = \phi_\otimes(n)\sum_{d|n}\frac{\phi(d)n_d}{d\phi_\otimes(n_d)} =$$

$$= \phi_\otimes(n)[\sum_{\substack{d|n\\ d\in 2\mathbb{Z}^+-1}} \frac{\phi(d)n_d}{d\phi_\otimes(n_d)} + \sum_{\substack{d|n\\ d\in 2\mathbb{Z}^+}} \frac{\phi(d)n_d}{d\phi_\otimes(n_d)}] =$$

$$= \phi_\otimes(n)[\tau(n) - \frac{1}{2}\tau_2(n)].$$

□

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