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On certain inequalities for σ, φ, ψ and related functions

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Abstract: Some new inequalities for the arithmetic functions of the title are considered. Among others we offer a refinement of a recent arithmetic inequality by K. T. Atanassov [1].
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1 Introduction

Let $\varphi(n)$, $\psi(n)$ and $\sigma(n)$ denote the classical arithmetic functions, representing Euler's totient, Dedekind's function, and the sum of divisors function respectively.

It is well-known that these functions are multiplicative, and for prime powers $n = p^a$ (p prime, $a \geq 1$ integer) one has

$$\varphi(p^a) = p^a \left(1 - \frac{1}{p}\right), \quad \psi(p^a) = p^a \left(1 + \frac{1}{p}\right), \quad \sigma(p^a) = \frac{p^{a+1} - 1}{p - 1}. \tag{1}$$

We have also by definition $\varphi(1) = \psi(1) = \sigma(1) = 1$.

In what follows, we shall need also the unitary analogues of the functions φ and σ; namely the arithmetical functions $\varphi^*(n)$ and $\sigma^*(n)$ (connected with the "unitary divisors" of n; see e.g. [2, 3] for many properties and references).

These functions are also multiplicative, and for prime powers they take the values

$$\varphi^*(p^a) = p^a - 1, \quad \sigma^*(p^a) = p^a + 1. \tag{2}$$

In a recent paper [1], K. T. Atanassov proved the following interesting inequality:

Theorem 1. *For all integers $n \geq 1$ we have the inequality*

$$\varphi(n)\psi(n)\sigma(n) \geq (n-1)(n+1)^2. \tag{3}$$

Remark 1. It is well-known that, $\varphi(n) \leq n-1$ for all $n \geq 2$ and $\psi(n) \geq n+1$, $\sigma(n) \geq n+1$. So relation (3) is not a consequence of known inequalities. For $n = p =$ prime there is equality.

In what follows, we shall prove the following refinement of (3) (so a new proof of (3) will be given, too):

Theorem 2. *For all $n \geq 1$ one has the inequalities*

$$\varphi(n)\psi(n)\sigma(n) \geq \varphi^*(n)(\sigma^*(n))^2 \geq (n-1)(n+1)^2. \tag{4}$$

There is equality in the first relation of (4) only when n is squarefree, or $n = 1$, while in the second one only when n is a prime power.

2 Proof of main result

For the first term of inequality (4), remark that both members are multiplicative functions. So, if $n = \prod_{i=1}^{r} p_i^{a_i}$ is the prime factorization of $n > 1$, it will be sufficient to prove the inequality for a prime power $p_i^{a_i}$. Then, the general result follows by a term-by-term multiplication of these inequality. Let for simplicity denote $p^a \equiv p_i^{a_i}$. Then we have to prove the relation (by using (1) and (2)):

$$p^{2a-2}(p+1)(p^{a+1}-1) \geq (p^a-1)(p^a+1)^2. \tag{5}$$

After elementary transformations, (5) may be written also as:

$$p^{3a-1} + p^a + 1 \geq p^{2a} + p^{2a-1} + p^{2a-2}. \tag{6}$$

We shall prove this inequality by induction upon $a \geq 1$. For $a = 1$, the relation is true (in fact, there is equality in (6)). Assuming (6) for a, let us try to prove it for $a+1$. By multiplying both sides of (6) by p^2, we get

$$p^{3a+1} + p^{a+2} + p^2 \geq p^{2a+2} + p^{2a+1} + p^{2a} = A,$$

and remark that A is in fact the right side of (6) for $a := a+1$. Therefore, it will be sufficient to prove that the left side of (6) for $a := a+1$ satisfies:

$$p^{3a+2} + p^{a+1} + 1 \geq p^{3a+1} + p^{a+2} + p^2. \tag{7}$$

This may be written also as

$$p^{3a+1}(p-1) \geq p^{a+1}(p-1) + p^2 - 1,$$

i.e.

$$p^{3a+1} \geq p^{a+1} + p + 1. \tag{8}$$

Now, inequality (8) is trivial, since equivalently states that

$$p^{a+1}(p^{2a} - 1) \geq p + 1,$$

and the left sides contains also

$$p^{2a} - 1 = (p^a + 1)(p - 1) \geq (p + 1)(p - 1) \geq p + 1$$

(the inequality is in fact strict).

Remark. The above proof shows in fact that, the inequality (6) is strict for $a > 1$. Thus one has equality in (5) only for $a = 1$, and this implies that there is equality for $n > 1$ in left side of (4) only when n is a product of distinct primes, i.e. $n =$ squarefree.

Now, the second inequality of (3), when

$$n = \prod_{i=1}^{r} p_i^{a_i} = \prod_{i=1}^{r} x_i > 1$$

can be rewritten as:

$$(x_1 - 1)\dots(x_r - 1)(x_1 + 1)^2 \dots (x_r + 1)^2 \geq (x_1 \dots x_r - 1)(x_1 \dots x_r + 1)^2, \tag{9}$$

where $r \geq 1$ and $x_i = p_i^{a_i}$. Clearly, there is equality in (9) for $r = 1$ (i.e., when n is a prime power); we shall prove that for $r > 1$ there is strict inequality.

First we prove the inequality for $r = 2$. The general case – via mathematical induction – will be reduced essentially to this case. Put for simplicity $x_1 = x$, $x_2 = y$ when the inequality becomes

$$(x - 1)(x + 1)^2(y - 1)(y + 1)^2 > (xy - 1)(xy + 1)^2. \tag{10}$$

Here $x \geq 2$ and $y \geq 3$ (as $p_1 \geq 2$, $p_2 \geq 3$ are distinct primes).

As $(x - 1)(x + 1)^2 = x^3 + x^2 - x - 1$, etc.; (10) may be written also as

$$(x^3 + x^2 - x - 1)(y^3 + y^2 - y - 1) > x^3y^3 + x^2y^2 - xy - 1,$$

or

$$x^3(y^2 - y - 1) + x^2(y^3 - y - 1) > x(y^3 + y^2 - 2y - 1) + y^3 + y^2 - y - 2. \tag{11}$$

Write this as

$$x[x(y^3 - y - 1) - (y^3 + y^2 - 2y - 1)] + x^3(y^2 - y - 1) > y^3 + y^2 - y - 2. \qquad (12)$$

Here

$$x(y^3 - y - 1) - (y^3 + y^2 - 2y - 1) \geq 2(y^3 - y - 1) - (y^3 + y^2 - 2y - 1)$$
$$= y^3 - y^2 - 1 > 0$$

by $x \geq 2$. Thus, the left side of (12) is

$$\geq 2(y^3 - y^2 - 1) + 8(y^2 - y - 1) > y^3 + y^2 - y - 2,$$

as this is

$$y^3 + 5y^2 - 7y - 8 > 0.$$

Now,

$$y(y^2 + 5y - 7) \geq 3(9 + 15 - 7) = 51 > 8,$$

and this proves (12), i.e. (10).

Now, assuming (9) for $r > 1$, let us try to prove it for $r + 1$; i.e.

$$(x_1 - 1)\dots(x_r - 1)(x_{r+1} - 1)(x_1 + 1)^2 \dots (x_r + 1)^2(x_{r+1} + 1)^2$$
$$> (x_1 \dots x_r x_{r+1} - 1)(x_1 \dots x_r x_{r+1} + 1)^2. \qquad (13)$$

By multiplying both sides of (9) with $(x_{r+1} - 1)(x_{r+1} + 1)^2$, it is sufficient to prove that

$$(x_1 \dots x_r - 1)(x_1 \dots x_r + 1)^2(x_{r+1} - 1)(x_{r+1} + 1)^2$$
$$> (x_1 \dots x_r x_{r+1} - 1)(x_1 \dots x_r x_{r+1} + 1)^2 \qquad (14)$$

Let $x_1 \dots x_r = x$, $x_{r+1} = y$. Then it is immediate that inequality (14) becomes exactly (10). This finishes the proof of Theorem 2. □

3 Notes and remarks

Remark 3. Other inequalities, connecting $\varphi^*(n)$ and $\sigma^*(n)$ were proved in [2] (in more general forms); for example

$$\frac{6}{\pi^2} \cdot n^2 < \varphi^*(n) \cdot \sigma^*(n) < n^2 \text{ for } n > 1, \qquad (15)$$

$$\varphi^*(n) + \sigma^*(n) \leq nd^*(n) \quad (n \geq 1), \qquad (16)$$

$$\varphi^*(n) + d^*(n) \leq \sigma^*(n) \quad (n \geq 1), \qquad (17)$$

$$d^*(n) \cdot n \leq \varphi^*(n)(d^*(n))^2 \leq n^2 \quad (n \geq 1), \qquad (18)$$

where $d^*(n) = 2^{\omega(n)}$ is the number of unitary divisors of n (here $\omega(n)$ denotes what is r in relation (9); i.e. the number of distinct prime factors of n).

Many new inequalities on the arithmetical functions $\varphi, \sigma, d, \varphi^*, \sigma^*, d^*$ are proved in our paper [4]. For example, we quote the relations:

$$\sigma^*(n) \leq d^*(n)\varphi(n) \text{ for any } n \geq 3 \text{ odd},$$

$$\sigma^*(n) \leq \frac{3}{2}d^*(n)\varphi(n) \text{ for } n \geq 2 \text{ even}. \tag{19}$$

It is easy to see that

$$\varphi(n) \leq \varphi^*(n),\ \sigma(n) \geq \sigma^*(n) \text{ and } d(n) \geq d^*(n) \text{ for } n \geq 1. \tag{20}$$

On the other hand, one has:

$$\sigma^*(n) \leq \varphi^*(n)(d^*(n))^\alpha,\ n \geq 1, \tag{21}$$

where $\alpha = \log_2 3$ (thus $1 < \alpha < 2$).

Clearly, inequalities $(15) - (18)$ or $(19) - (21)$ may be connected with relation (4). The right side of (15) implies

$$n^2\sigma^*(n) > \varphi^*(n)(\sigma^*(n))^2 \geq (n-1)(n+1)^2 \quad (n > 1). \tag{22}$$

Another example is

$$\varphi^*(n)(d^*(n))^2(\varphi(n))^2 \geq \varphi^*(n)(\sigma^*(n))^2 \geq (n-1)(n+1)^2 \text{ for } n \geq 3 \text{ odd}, \tag{23}$$

which is a consequence of (19) and (4), etc.

Remark 4. In paper [5], it is proved that

$$\sigma(n) > n + (\omega(n) - 1)\sqrt{n} \text{ for } n \geq 2. \tag{24}$$

As an application of (24), it is shown that

$$\sigma(n) > n + \sqrt{n} \text{ if and only if } n \neq \text{prime}, \tag{25}$$

$$\sigma(n) > n + \sqrt{n} + \sqrt[3]{n} \text{ if and only if } n \neq \text{prime and } n \neq (\text{prime})^2. \tag{26}$$

It is immediate that, $\sigma(n) \geq \psi(n)$ for any $n \geq 1$. In paper [6], it is shown that

$$\sigma(n) < \frac{\pi^2}{6} \cdot \psi(n) \text{ for } n \geq 1. \qquad (26)$$

As $\frac{\pi^2}{6} < 1,7 < 2$, particularly we get $\sigma(n) < 2\psi(n)$. A stronger inequality than this last one – which is however not comparable with (26) – is due to Ch. Wall (see [3]):

$$\psi(n) \geq \frac{\sigma(n) + \sigma^*(n)}{2}. \tag{27}$$

By (20) we get

$$\sigma(n) \geq \psi(n) \geq \frac{\sigma(n) + \sigma^*(n)}{2} \geq \sigma^*(n) \tag{28}$$

which particularly shows that, $\psi(n)$ lies between $\sigma^*(n)$ and $\sigma(n)$.

4 Related results

By using relations (1) and (2), one can deduce the following formulae:

$$\varphi(n)\sigma(n) = n^2 \cdot \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^{a_i+1}}\right), \tag{29}$$

$$\varphi^*(n)\sigma^*(n) = n^2 \cdot \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^{2a_i}}\right), \tag{30}$$

$$\varphi(n)\psi(n) = n^2 \cdot \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^2}\right), \tag{31}$$

where $n = \prod_{i=1}^{r} p_i^{a_i}$ is the prime factorization of $n > 1$.

Theorem 3. *For all $n > 1$ one has*

$$\varphi(n)\psi(n) \leq \varphi(n)\sigma(n) \leq \varphi^*(n)\sigma^*(n) \leq n^2 - 1, \tag{32}$$

$$\varphi(n)\sigma(n) \leq n^2 - \frac{n}{\gamma(n)} \leq n^2 - 1, \tag{33}$$

$$\varphi(n)\psi(n) \leq n^2 - \left(\frac{n}{\gamma(n)}\right)^2 \leq n^2 - 1, \tag{34}$$

where $\gamma(n) = \prod_{i=1}^{r} p_i$ denotes the "core of n" (see e.g. [3] for this function).

Proof. As $2 \leq a_i + 1 \leq 2a_i$, the first two inequalities of (32) are consequences of relations $(29) - (31)$. For the last inequality of (32) use the classical (Weierstrass-type) inequality:

$$(x_1 - 1)(x_2 - 1) \ldots (x_r - 1) \leq x_1 x_2 \ldots x_r - 1, \tag{35}$$

where $r \geq 1$ is integer, and $x_i > 1$ $(i = 1, 2, \ldots, r)$ are arbitrary real numbers. Apply now (35)

for $x_i = p_i^{2a_i}$ in order to deduce the last inequality of (32).

Applying (35) for $x_i = p_i^{a_i+1}$, we get the first inequality of (33). By $n \geq \gamma(n)$, clearly the last relation of (33) follows, too. Finally, apply (35) for $x_i = p_i^2$ for the proof of (34). □

Theorem 4. *For any $n > 1$, the following refinement of (26) holds true:*

$$\frac{\psi(n)}{\sigma(n)} > \frac{\varphi(n)\psi(n)}{n^2} > \frac{6}{\pi^2}. \tag{36}$$

Proof. The first inequality of (36) follows by $\varphi(n)\sigma(n) < n^2$, which is contained particularly in (32). For the second inequality of (36) remark that by (31),

$$\varphi(n)\psi(n) = n^2 \cdot \prod_{i=1}^{n} \left(1 - \frac{1}{p_i^2}\right) > n^2 \cdot \prod_{p\ prime} \left(1 - \frac{1}{p^2}\right),$$

where p runs through the set of all prime numbers. It is well-known, from the Euler product representation of the zeta function that

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{n^k} = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Particularly,

$$\zeta(2) = \prod_{p\ prime} \left(1 - \frac{1}{p^2}\right)^{-1}, \quad \text{i.e.} \quad \prod_{p\ prime} \left(1 - \frac{1}{p^2}\right) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}$$

by the Euler series $\sum_{k=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

This proves the second inequality of (36). □

Remark 5. By (32), the right side of (36) offers also a strong refinement of left side of (15).

The lower bound from the second inequality of (36) is best possible in a sense, since there exists a sequence (n_k) such that

$$\lim_{k\to\infty} \frac{\varphi(n_k)\psi(n_k)}{n_k^2} = \frac{6}{\pi^2},$$

namely $n_k = p_1 p_2 \dots p_k$, where p_k now is the kth prime number.

For certain particular values of n, however, better lower bounds will be provided by:

Theorem 5. *Let $p(n)$, resp. $P(n)$ denote the least, resp. largest prime factors of n. Then*

$$\frac{\varphi(n)\psi(n)}{n^2} \geq \left(1 - \frac{1}{p(n)}\right)\left(1 + \frac{1}{P(n)}\right). \tag{37}$$

Proof. We will use the remark that

$$\frac{p_i+1}{p_i} \geq \frac{p_{i+1}}{p_{i+1}-1}, \text{ for } i = 1, 2, \ldots, k-1 \tag{38}$$

where $2 \leq p_1 < p_2 < \ldots < p_k$ are the distinct prime factors of n.

Indeed, (38) is in fact $p_{i+1} - p_i \geq 1$. Now, by

$$\frac{\psi(n)}{n} = \frac{p_1+1}{p_1} \cdot \frac{p_2+1}{p_2} \cdots \frac{p_{k-1}+1}{p_{k-1}} \cdot \frac{p_k+1}{p_k}$$

and (38), we can write

$$\frac{\psi(n)}{n} \geq \frac{p_2}{p_2-1} \cdots \frac{p_k}{p_k-1} \cdot \frac{p_k+1}{p_k} = \frac{p_1-1}{p_1} \cdot \frac{p_k+1}{p_k} \left(\frac{p_1}{p_1-1} \cdots \frac{p_k}{p_k-1} \right),$$

where the parenthesis is in fact $\frac{n}{\varphi(n)}$.

Since $p_1 = p(n)$, $p_k = P(n)$, inequality (37) follows. □

Corollary. *If* $n \geq 3$ *is odd, then*

$$\frac{\varphi(n)\psi(n)}{n^2} \geq \frac{2}{3}\left(1 + \frac{1}{P(n)}\right) > \frac{2}{3} > \frac{6}{\pi^2}. \tag{39}$$

Proof. Since $1 - \frac{1}{p(n)} \geq 1 - \frac{1}{3}$ (by $p(n) \geq 3$), from (37) we get the first inequality. The last inequality holds, as $\pi^2 > 9$. □

Remark 6. If $n \geq 2$ even, we get

$$\frac{\varphi(n)\psi(n)}{n^2} \geq \frac{1}{2}\left(1 + \frac{1}{P(n)}\right). \tag{40}$$

The right side of (40) is $> \frac{6}{\pi^2}$ only if $P(n) < \frac{\pi^2}{12-\pi^2} = 4.6, \ldots$, so $P(n) \leq 3$, i.e., when n is of the form $n = 2^a \cdot 3^b$ ($a \geq 1$, $b \geq 0$ integers).

If $n \geq 3$ is odd, not divisible by 3, then (39) may be refined

$$\frac{\varphi(n)\psi(n)}{n^2} \geq \frac{4}{5}\left(1 + \frac{1}{P(n)}\right) > \frac{4}{5} > \frac{2}{3}. \tag{41}$$

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