Notes on Number Theory and Discrete Mathematics
ISSN 1310–5132
Vol. 20, 2014, No. 2, 44–51

Mean values of the error term with shifted arguments in the circle problem

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Abstract: In this paper, we show the relation between the shifted sum of a number-theoretic error term and its continuous mean (integral). We shall obtain a certain expression of the shifted sum as a linear combination of the continuous mean with the Bernoulli polynomials as their coefficients. As an application of our theorem, we give better approximations of the continuous mean by a shifted sum.
Keywords: The circle problem, Mean value of error terms, Shifted sum, Bernoulli polynomial.
AMS Classification: 11N37.

1 Introduction

Let $f(n)$ be an arithmetical function and let $E(x)$ be the number-theoretic error term defined by

$$E(x) = \sum_{n \le x} f(n) - g(x), \tag{1.1}$$

where $g(x)$ is the main term of the summatory function of $f(n)$, which is usually written by infinitely differentiable elementary functions. When $f(n) = d(n)$, the number of positive divisors of n, then $g(x) = x(\log x + 2\gamma - 1)$ (γ being the Euler constant) and $E(x)$ is usually denoted by $\Delta(x)$. On the other hand when $f(n) = r(n)$, the number of ways to write n as a sum of two squares of integers, then $g(x) = \pi x$ and $E(x)$ is usually denoted by $P(x)$. There are a lot of researches for $E(x)$; the upper bound estimate, the asymptotic behavior of the "continuous mean" $\int_1^x E(t)^k dt$ and the "discrete mean" $\sum_{n\le x} E(n)^k$, etc. In particular, we studied the difference of these two kinds of mean values for $E(x) = \Delta(x)$, $P(x)$ and the error term in the case of Rankin-Selberg series [1, 3, 4].

In our previous paper [2], we derived a certain kind of expressions of a shifted sum of $\Delta(x)$. Here we call a shifted sum of $\Delta(x)$ as a sum of $\Delta(n+\alpha)^k$ over $n \le x$ (see (1.4) below). For example, in the fourth power case we proved that

$$\sum_{n\le x} \Delta(n+\alpha)^4 = \int_1^x \Delta(t)^4 dt + B_1(\alpha) x^{7/4}(A_1 \log x + A_2) + B_2(\alpha) x^{3/2}(A_3 \log^2 x + A_4 \log x + A_5) + O(x^{7/5+\varepsilon}) \tag{1.2}$$

holds with some suitable constants A_j. Here $B_n(x)$ is the nth Bernoulli polynomial defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n.$$

For example, $B_1(x) = x - \frac{1}{2}$, $B_2(x) = x^2 - x + \frac{1}{6}$ and $B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$. The feature of this formula is that the difference between the shifted sum and the integral is expressed as a linear combination of terms of lower degrees of x with the Bernoulli polynomials as the coefficients. Moreover, we can see that by taking the special α, for example $\alpha = 1/2$, the formula (1.2) gives better approximation to each other than the previous case $\alpha = 0$ [1].

In this paper we shall study the reason why Bernoulli polynomials appear in the shifted sum. For the sake of simplicity we assume, in this paper,

$$g(x) = Ax \tag{1.3}$$

with a certain constant A. There are many arithmetical functions with this property. The most important one is $r(n)$, in which case $A = \pi$. Other examples are $f(n) = \varphi(n)/n$, where $\varphi(n)$ is the Euler totient function, $f(n) = \sigma(n)/n$, where $\sigma(n)$ is the sum of positive divisors of n and $f(n) = c(n)$, where $c(n)$ is the coefficient of Rankin-Selberg series (for the last example see [1]).

To state our theorem, we shall introduce some notations. Let $x > 2$ be a real number and $0 \le \alpha < 1$. We define the shifted sum by

$$D_k(x, \alpha) = \sum_{n\le x} E(n+\alpha)^k. \tag{1.4}$$

We write $D_k(x) = D_k(x, 0)$ for short. The "continuous mean value" is customarily formulated by means of the integral $I_k(x)$ (for this definition, see (3.2) below). But, for our purpose, it is convenient to introduce the "modified mean value"

$$\widetilde{I}_k(x) = \int_1^{[x]+1} E(t)^k dt, \tag{1.5}$$

where $[x]$ is the largest integer not exceeding x. It is important for our theorem to use $\widetilde{I}_k(x)$ instead of $I_k(x)$. We study the relation between $D_k(x,\alpha)$ and $\widetilde{I}_k(x)$ and obtain the following

Theorem. *Let $E(x)$ be the function defined by* (1.1). *If $g(x)$ satisfies* (1.3)*, then we have*

$$D_k(x,\alpha) = \sum_{j=0}^{k} \binom{k}{j} (-A)^{k-j} B_{k-j}(\alpha) \widetilde{I}_j(x). \tag{1.6}$$

Especially we have

$$D_k(x) = \sum_{j=0}^{k} \binom{k}{j} (-A)^{k-j} B_{k-j} \widetilde{I}_j(x),$$

where $B_n = B_n(0)$ denotes the nth Bernoulli number.

Based on this Theorem, we shall give sharper estimates of the difference between the shifted sum $D_k(x,\alpha)$ and the "continuous mean value" $I_k(x)$ in Section 3.

As in [2], it is also possible to give an interpretation of our Theorem in terms of Dirichlet series whose coefficients are $P(n+\alpha)^k$, but we shall omit it in this paper (cf. [5, 6]).

2 Proof of Theorem

We make use of the method of generating functions. So let X be an indeterminate variable in this section.

Lemma 1. *Suppose that $g(x)$ satisfies* (1.3). *Then we have*

$$\sum_{k=0}^{\infty} \frac{D_k(x,\alpha)}{k!} X^k = e^{-A\alpha X} \sum_{k=0}^{\infty} \frac{D_k(x)}{k!} X^k. \tag{2.1}$$

Proof. Since $g(x) = Ax$, we have

$$E(n+\alpha) = E(n) - A\alpha.$$

Hence

$$D_k(x,\alpha) = \sum_{n\le x} (E(n) - A\alpha)^k = \sum_{n\le x} \sum_{j=0}^{k} \binom{k}{j} E(n)^j (-A\alpha)^{k-j}.$$

Interchanging the sums over n and j, we have

$$\frac{D_k(x,\alpha)}{k!} = \sum_{j=0}^{k} \frac{(-A\alpha)^{k-j}}{(k-j)!} \frac{D_j(x)}{j!}. \tag{2.2}$$

Since $e^{-A\alpha X} = \sum_{m=0}^{\infty} \frac{(-A\alpha)^m}{m!} X^m$, the right-hand side of (2.2) coincides with the coefficient of X^k of the right-hand side of (2.1). This completes the proof of the lemma. □

Lemma 2. *Let* $\widetilde{I}_k(x)$ *be the function defined by* (1.5). *Then we have*

$$\sum_{k=0}^{\infty}\frac{\widetilde{I}_k(x)}{k!}X^k=\frac{e^{-AX}-1}{-AX}\sum_{k=0}^{\infty}\frac{D_k(x)}{k!}X^k. \tag{2.3}$$

Proof. In order to prove (2.3), we recall a formula proved in our previous paper [1]. In Lemma 1 of [1], we showed that (in the notation there)

$$\sum_{n\le x}E(n)^k-\int_1^x E(t)^k dt=\sum_{j=0}^{k-1}\binom{k}{j}(-1)^{k-j+1}\sum_{n\le x}E(n)^j\int_n^{n+1}(g(t)-g(n))^{k-j}dt$$
$$+\int_x^{[x]+1}E(t)^k dt. \tag{2.4}$$

We specialize $g(x)=Ax$ in (2.4). Then (2.4) implies that

$$\widetilde{I}_k(x)=\sum_{j=0}^{k}\binom{k}{j}\frac{(-A)^{k-j}}{k-j+1}D_j(x). \tag{2.5}$$

Writing (2.5) in the form

$$\frac{\widetilde{I}_k(x)}{k!}=\sum_{j=0}^{k}\frac{(-A)^{k-j}}{(k-j+1)!}\frac{D_j(x)}{j!}$$

and noting

$$\sum_{k=0}^{\infty}\frac{(-A)^k}{(k+1)!}X^k=\frac{e^{-AX}-1}{-AX},$$

we get the equality of (2.3). □

Proof of Theorem. By (2.1) and (2.3), we have

$$\sum_{k=0}^{\infty}\frac{D_k(x,\alpha)}{k!}X^k=\frac{-AX}{e^{-AX}-1}\cdot e^{-A\alpha X}\sum_{k=0}^{\infty}\frac{\widetilde{I}_k(x)}{k!}X^k$$
$$=\sum_{k=0}^{\infty}\frac{B_k(\alpha)}{k!}(-AX)^k\sum_{k=0}^{\infty}\frac{\widetilde{I}_k(x)}{k!}X^k. \tag{2.6}$$

Here we used the definition of Bernoulli polynomials. Now comparing the coefficients of X^k of (2.6), we get the equality (1.6). □

It is instructive to write up the identities of Theorem for small k:

$$D_1(x,\alpha)=\widetilde{I}_1(x)-AB_1(\alpha)[x],$$
$$D_2(x,\alpha)=\widetilde{I}_2(x)-2AB_1(\alpha)\widetilde{I}_1(x)+A^2B_2(\alpha)[x],$$
$$D_3(x,\alpha)=\widetilde{I}_3(x)-3AB_1(\alpha)\widetilde{I}_2(x)+3A^2B_2(\alpha)\widetilde{I}_1(x)-A^3B_3(\alpha)[x],$$
$$D_4(x,\alpha)=\widetilde{I}_4(x)-4AB_1(\alpha)\widetilde{I}_3(x)+6A^2B_2(\alpha)\widetilde{I}_2(x)-4A^3B_3(\alpha)\widetilde{I}_1(x)$$
$$+A^4B_4(\alpha)[x],$$
$$D_5(x,\alpha)=\widetilde{I}_5(x)-5AB_1(\alpha)\widetilde{I}_4(x)+10A^2B_2(\alpha)\widetilde{I}_3(x)-10A^3B_3(\alpha)\widetilde{I}_2(x)$$
$$+5A^4B_4(\alpha)\widetilde{I}_1(x)-A^5B_5(\alpha)[x].$$

We note that the formula (1.6) can be regarded as a kind of inversion formula. It may be interesting from the viewpoint on combinatorial theory.

3 Mean values of the function related with $P(x)$

In this section we consider the case of the Gauss circle problem. Hence $f(n) = r(n)$ defined in Introduction. Then $g(x) = \pi x$ and $E(x) = P(x)$.

The upper bound of $P(x)$ has been studied for a long time. Let λ_0 be the number defined by

$$\lambda_0 = \inf\{\lambda \mid P(x) = O(x^\lambda)\}. \tag{3.1}$$

The first non-trivial result is $\lambda_0 \leq 1/3$, which is due to Sierpiński in 1906. It is known that $\lambda_0 \leq (k+l)/(2k+2)$, where (k,l) is any exponent pair (see Graham and Kolesnik [7]). For instance, the exponent pair $(k,l) = (97/251, 132/251)$ gives $\lambda_0 \leq 229/696 = 0.32902\ldots$. The best estimate up to now is $\lambda_0 \leq 131/416 = 0.3149\ldots$ due to Huxley [8]. It is known that $\lambda_0 \geq 1/4$ and is conjectured that $\lambda_0 = 1/4$. For more details for $P(x)$, see Graham and Kolesnik [7] and Krätzel [11].

As stated in Introduction, the "continuous mean value" estimate is usually formulated by means of the integral

$$I_k(x) = \int_1^x P(t)^k dt. \tag{3.2}$$

Note that the upper limit of $I_k(x)$ is x, while that of $\widetilde{I}_k(x)$ is $[x]+1$, hence

$$\widetilde{I}_k(x) - I_k(x) = \int_x^{[x]+1} P(t)^k dt = O(x^{k\lambda_0+\varepsilon}), \tag{3.3}$$

where λ_0 is defined by (3.1). By this trivial estimate we can replace the terms $\widetilde{I}_k(x)$ in Theorem by $I_k(x)$.

We shall recall some basic results on this integral:

$$I_1(x) = -x - \frac{x^{3/4}}{\pi^2}\sum_{n=1}^{\infty}\frac{r(n)}{n^{5/4}}\sin(2\pi\sqrt{nx} + \frac{\pi}{4}) + O(x^{1/4}), \tag{3.4}$$

and

$$I_k(x) = C_k x^{1+k/4} + Q_k(x) \tag{3.5}$$

for $2 \leq k \leq 9$, where C_k are certain positive constants and $Q_k(x)$ are error terms. Recently Lau and Tsang proved that $Q_2(x) = O(x \log x \log\log x)$ [13]. For $3 \leq k \leq 9$, $Q_k(x) = O(x^{\rho_k+\varepsilon})$ are known with $\rho_3 = 7/5, \rho_4 = 53/28, \rho_5 = 177/80, \rho_6 = 5910/2371, \rho_7 = 17341/6312, \rho_8 = 28291/9433$ and $\rho_9 = 244439/75216$ [10, 14, 15, 16]. In their paper [12], Lau and Tsang studied the error term of the mean square in the case of the Dirichlet divisor problem and proposed a conjecture on the behavior of this error term. Though they did not mention explicitly the corresponding conjecture in the case of the circle problem, it is plausible to conjecture that

$$Q_2(x) = cx\log x + O(x) \tag{3.6}$$

for some constant c. (It may be possible that c is zero.)

For much higher cases $k \geq 10$, no asymptotic estimates for $I_k(x)$ are known. However, it is known that

$$I_K(x) \ll x^{\frac{35K+38}{108}+\varepsilon} \quad \text{for any real } K \geq 35/4, \tag{3.7}$$

(see Ivić [9, Theorem 13.12]).

As we stated in Introduction, the difference $D_k(x) - I_k(x)$ was treated in [1, 3, 4] in details. These formulas can be regarded as the approximations of $I_k(x)$ by means of $D_k(x)$, From our Theorem, we get similar approximations of $I_k(x)$ by means of the shifted sum as in [2]. In fact, by (1.6), (3.4), (3.5) and (3.3) we get the following corollary.

Corollary 1. *Let α be any real number such that $0 \leq \alpha < 1$. Then we have*

$$\begin{aligned}
D_1(x,\alpha) &= I_1(x) - \pi B_1(\alpha)x + O(x^{\lambda_0+\varepsilon}),\\
D_2(x,\alpha) &= I_2(x) + \left(2\pi B_1(\alpha) + \pi^2 B_2(\alpha)\right) x + O(x^{3/4}),\\
D_3(x,\alpha) &= I_3(x) - 3\pi B_1(\alpha) C_2 x^{3/2} + O(x \log x \log\log x),\\
D_4(x,\alpha) &= I_4(x) - 4\pi B_1(\alpha) C_3 x^{7/4} + 6\pi^2 B_2(\alpha) C_2 x^{3/2} + O(x^{7/5+\varepsilon}),\\
D_k(x,\alpha) &= I_k(x) - k\pi B_1(\alpha) C_{k-1} x^{(k+3)/4} + O(x^{\rho_{k-1}+\varepsilon})
\end{aligned} \tag{3.8}$$

for $5 \leq k \leq 10$. *If we assume* (3.6) *we have*

$$D_3(x,\alpha) = I_3(x) - 3\pi B_1(\alpha) C_2 x^{3/2} - 3\pi B_1(\alpha) c x \log x + O(x).$$

If ρ_k $(4 \leq k \leq 9)$ are improved, then the estimate of (3.8) may be improved automatically. We note that when $\alpha = 0$ these results are already obtained by [3, 1].

If we specialize the value α, we may get better approximations of $I_k(x)$ by $D_k(x,\alpha)$. In fact it is easily seen that if we set $\alpha = 1/2$, the second term on the right-hand side of each formula of Corollary 1 becomes zero. More precisely we have

Corollary 2.

$$\begin{aligned}
D_1(x,1/2) &= I_1(x) + O(x^{\lambda_0+\varepsilon}),\\
D_2(x,1/2) &= I_2(x) - \frac{\pi^2}{12} x + O(x^{2\lambda_0+\varepsilon}),\\
D_3(x,1/2) &= I_3(x) + \frac{\pi^2}{4} x + O(x^{3\lambda_0+\varepsilon}),\\
D_4(x,1/2) &= I_4(x) - \frac{\pi^2}{2} C_2 x^{3/2} + O(x^{4\lambda_0+\varepsilon}).
\end{aligned}$$

For $5 \leq k \leq 11$ *we have*

$$D_k(x,1/2) = I_k(x) - \frac{k(k-1)\pi^2}{24} C_{k-2} x^{\frac{k+2}{4}} + O(x^{k\lambda_0+\varepsilon}) + O(x^{\rho_{k-2}+\varepsilon}).$$

Proof. Noting $B_1(1/2) = 0$ and $B_3(1/2) = 0$, the expressions $D_k(x,1/2)$ are obtained directly for $k = 1, 2$ and 3. For $4 \leq k \leq 11$, we have

$$D_k(x,1/2) = \widetilde{I}_k(x) + \binom{k}{2} \pi^2 B_2(1/2) \widetilde{I}_{k-2}(x) + O(|\widetilde{I}_{k-4}(x)|).$$

We note that

$$
\begin{aligned}
\widetilde{I}_{k-2}(x) &= I_{k-2}(x) + O(x^{(k-2)\lambda_0+\varepsilon}) \\
&= C_{k-2}x^{\frac{k+2}{4}} + O(x^{\rho_{k-2}+\varepsilon}) + O(x^{(k-2)\lambda_0+\varepsilon}).
\end{aligned}
$$

Hence we have

$$D_k(x,1/2) = I_k(x) - \frac{k(k-1)\pi^2}{24}C_{k-2}x^{\frac{k+2}{4}} + O(x^{k\lambda_0+\varepsilon}) + O(x^{\rho_{k-2}+\varepsilon}).$$

□

In order to get better approximations, we may consider the average of the shifted sum like

$$T_k(x) = \sum_{n\le x}\frac{P(n+\beta)^k + P(n+\beta')^k}{2}, \tag{3.9}$$

where β and β' are the roots of the equation $B_2(x) = 0$ as we have already considered in [2]. Since

$$B_j(\beta) + B_j(\beta') = 0$$

for $j = 1, 2$ and 3 simultaneously, we have the following approximation.

Corollary 3. *Let* $T_k(x)$ *be the function defined by* (3.9). *Then we have*

$$T_k(x) = I_k(x) + O(x^{k\lambda_0+\varepsilon}) \tag{3.10}$$

for $2 \le k \le 13$ *and*

$$T_k(x) = I_k(x) + O(x^{k\lambda_0+\varepsilon}) + O(x^{\frac{35k-102}{108}+\varepsilon}) \tag{3.11}$$

for $k \ge 14$.

Proof. For $k \le 5$, the estimates (3.10) are obtained directly. For $k \ge 6$, we have

$$T_k(x) = I_k(x) + O(x^{k\lambda_0+\varepsilon}) + O(|\widetilde{I}_{k-4}(x)|). \tag{3.12}$$

But we have $|\widetilde{I}_{k-4}(x)| \ll x^{k/4} \ll x^{k\lambda_0}$ for $k \le 13$, hence we get (3.10). The formula (3.11) follows from (3.12) and (3.7). □

Finally we should note that the shifted sums $D_k(x,\alpha)$ and $T_k(x)$ can be regarded as better approximations for $I_k(x)$ by the formulas in Corollaries 1–3.

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