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On two Diophantine equations $2A^6 + B^6 = 2C^6 \pm D^3$

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Abstract: We give parametric solutions, and thus show that the two Diophantine equations $2A^6 + B^6 = 2C^6 \pm D^3$ have infinitely many nontrivial and primitive solutions in positive integers (A, B, C, D).
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In the loving memory of my grandparents!

1 Introduction

There is extensive study on the Diophantine equation

$$X_1^6 + X_2^6 + X_3^6 = Y_1^6 + Y_2^6 + Y_3^6, \tag{1.1}$$

and many papers (see [1]–[6]) dealing with different aspects of (1.1) have appeared in journals. But, the pair of Diophantine equations

$$2X_1^6 + X_2^6 = 2Y_1^6 \pm Y_2^6 \tag{1.2}$$

have not yet been investigated. Hence, in this paper, we study two similar Diophantine equations

$$2A^6 + B^6 = 2C^6 \pm D^3, \tag{1.3}$$

which may raise some hope in dealing with (1.2). Based on an elementary approach, we obtain some parametric solutions for (1.3).

2 Parameterising $2A^6 + B^6 = 2C^6 \pm D^3$

We need the following Lemma for parameterising (1.3).

Lemma 2.1. *For any real values of a and b there is a polynomial identity*

$$(a^2+ab-b^2)^2-(a^2+ab-b^2)(a^2-ab-b^2)+(a^2-ab-b^2)^2=(a^4+a^2b^2+b^4). \quad (2.1)$$

Proof. Let us expand and simplify the LHS of (2.1).

$$(a^2+ab-b^2)^2=(a^4+2a^3b-a^2b^2-2ab^3+b^4); \quad (2.2)$$
$$(a^2+ab-b^2)(a^2-ab-b^2)=(a^4-3a^2b^2+b^4); \quad (2.3)$$
$$(a^2-ab-b^2)^2=(a^4-2a^3b-a^2b^2+2ab^3+b^4). \quad (2.4)$$

Using (2.2), (2.3) and (2.4) we get

$$\begin{aligned}\text{LHS of (2.1)} &= (a^4+2a^3b-a^2b^2-2ab^3+b^4)-(a^4-3a^2b^2+b^4)\\ &\quad +(a^4-2a^3b-a^2b^2+2ab^3+b^4);\\ &= (a^4+2a^3b-a^2b^2-2ab^3+b^4-a^4+3a^2b^2-b^4\\ &\quad +a^4-2a^3b-a^2b^2+2ab^3+b^4);\\ &= (a^4+a^2b^2+b^4)=\text{RHS of (2.1)}.\end{aligned}$$

Hence, the proof is complete. □

Now, we have

$$\begin{aligned}&(a^2+ab-b^2)^3+(a^2-ab-b^2)^3=\{(a^2+ab-b^2)+(a^2-ab-b^2)\}\times \quad (2.5)\\ &\{(a^2+ab-b^2)^2-(a^2+ab-b^2)(a^2-ab-b^2)+(a^2-ab-b^2)^2\}\\ &=2(a^2-b^2)(a^4+a^2b^2+b^4)[\text{by (2.1)}]=2(a^6-b^6).\end{aligned}$$

From (2.5) we get

$$2b^6+(a^2+ab-b^2)^3=2a^6-(a^2-ab-b^2)^3. \quad (2.6)$$

In (2.6) take

$$a^2+ab-b^2=c^2. \quad (2.7)$$

By (2.6) and (2.7) we get

$$2b^6+c^6=2a^6-(a^2-ab-b^2)^3. \quad (2.8)$$

From (2.7) we have

$$\begin{aligned}&a^2+ab-b^2-c^2=0;\\ \Rightarrow\ &a=\{-b\pm\sqrt{(b^2+4b^2+4c^2)}\}/2;\\ \Rightarrow\ &a=\{-b\pm\sqrt{(5b^2+4c^2)}\}/2. \quad (2.9)\end{aligned}$$

In (2.9) take

$$d^2 = 5b^2 + 4c^2. \tag{2.10}$$

By (2.9) and (2.10) we get

$$a = (-b \pm d)/2. \tag{2.11}$$

From (2.10) we get

$$d^2 - 4c^2 = 5b^2; \Rightarrow (d + 2c)(d - 2c) = 5b^2. \tag{2.12}$$

In (2.12) take

$$b = b_1 b_2; \ (d + 2c) = 5b_1^2; \text{ and } (d - 2c) = b_2^2. \tag{2.13}$$

Now, solving for d and c we get

$$d = (5b_1^2 + b_2^2)/2; \tag{2.14}$$

and

$$c = (5b_1^2 - b_2^2)/4. \tag{2.15}$$

In (2.11), substituting b and d from (2.13) and (2.14) we get

$$\begin{aligned} a &= (-b_1 b_2 \pm (5b_1^2 + b_2^2)/2)/2; \\ \Rightarrow a &= (-2b_1 b_2 \pm (5b_1^2 + b_2^2))/4. \end{aligned} \tag{2.16}$$

In (2.8), take $a = (5b_1^2 - 2b_1 b_2 + b_2^2)/4$, $b = b_1 b_2$ and $c = (5b_1^2 - b_2^2)/4$ from (2.16), (2.13) and (2.15) respectively to get

$$\begin{aligned} 2(b_1 b_2)^6 + \{(5b_1^2 - b_2^2)/4\}^6 &= 2\{(5b_1^2 - 2b_1 b_2 + b_2^2)/4\}^6 \\ &- \{((5b_1^2 - 2b_1 b_2 + b_2^2)/4)^2 - ((5b_1^2 - 2b_1 b_2 + b_2^2)/4)b_1 b_2 - (b_1 b_2)^2\}^3. \end{aligned} \tag{2.17}$$

Multiplying both the sides of (2.17) by 4^6, and simplifying, we get

$$\begin{aligned} 2(4b_1 b_2)^6 + (5b_1^2 - b_2^2)^6 &= 2(5b_1^2 - 2b_1 b_2 + b_2^2)^6 \\ &- \{(5b_1^2 - 2b_1 b_2 + b_2^2)^2 - 4(5b_1^2 - 2b_1 b_2 + b_2^2)b_1 b_2 - (4b_1 b_2)^2\}^3; \\ \Rightarrow 2(4b_1 b_2)^6 + (5b_1^2 - b_2^2)^6 &= 2(5b_1^2 - 2b_1 b_2 + b_2^2)^6 \\ &- (25b_1^4 - 40b_1^3 b_2 + 6b_1^2 b_2^2 - 8b_1 b_2^3 + b_2^4)^3. \end{aligned} \tag{2.18}$$

Similarly in (2.8), take $a = (-5b_1^2 - 2b_1 b_2 - b_2^2)/4$, $b = b_1 b_2$ and $c = (5b_1^2 - b_2^2)/4$ from (2.16), (2.13) and (2.15) respectively to get

$$\begin{aligned} 2(b_1 b_2)^6 + \{(5b_1^2 - b_2^2)/4\}^6 &= 2\{(-5b_1^2 - 2b_1 b_2 - b_2^2)/4\}^6 \\ &- \{((-5b_1^2 - 2b_1 b_2 - b_2^2)/4)^2 - ((-5b_1^2 - 2b_1 b_2 - b_2^2)/4)b_1 b_2 - (b_1 b_2)^2\}^3; \\ \Rightarrow 2(b_1 b_2)^6 + \{(5b_1^2 - b_2^2)/4\}^6 &= 2\{(5b_1^2 + 2b_1 b_2 + b_2^2)/4\}^6 \\ &- \{((5b_1^2 + 2b_1 b_2 + b_2^2)/4)^2 + ((5b_1^2 + 2b_1 b_2 + b_2^2)/4)b_1 b_2 - (b_1 b_2)^2\}^3. \end{aligned} \tag{2.19}$$

Multiplying both the sides of (2.19) by 4^6, and simplifying, we get

$$\begin{aligned}&2(4b_1b_2)^6 + (5b_1^2 - b_2^2)^6 = 2(5b_1^2 + 2b_1b_2 + b_2^2)^6\\ &\quad - \{(5b_1^2 + 2b_1b_2 + b_2^2)^2 + 4(5b_1^2 + 2b_1b_2 + b_2^2)b_1b_2 - (4b_1b_2)^2\}^3;\\ \Rightarrow &2(4b_1b_2)^6 + (5b_1^2 - b_2^2)^6 = 2(5b_1^2 + 2b_1b_2 + b_2^2)^6 \qquad (2.20)\\ &\quad - (25b_1^4 + 40b_1^3b_2 + 6b_1^2b_2^2 + 8b_1b_2^3 + b_2^4)^3.\end{aligned}$$

In (2.18), taking $b_2 = (b_1 + 1)$ we get

$$\begin{aligned}&2\{4b_1(b_1+1)\}^6 + \{5b_1^2 - (b_1+1)^2\}^6 = 2\{5b_1^2 - 2b_1(b_1+1) + (b_1+1)^2\}^6\\ &\quad - \{25b_1^4 - 40b_1^3(b_1+1) + 6b_1^2(b_1+1)^2 - 8b_1(b_1+1)^3 + (b_1+1)^4\}^3;\\ \Rightarrow &2(4b_1(b_1+1))^6 + (4b_1^2 - 2b_1 - 1)^6 = 2(4b_1^2+1)^6\\ &\quad - (-16b_1^4 - 48b_1^3 - 12b_1^2 - 4b_1 + 1)^3;\\ \Rightarrow &2(4b_1(b_1+1))^6 + (4b_1^2 - 2b_1 - 1)^6 = 2(4b_1^2+1)^6 \qquad (2.21)\\ &\quad + (16b_1^4 + 48b_1^3 + 12b_1^2 + 4b_1 - 1)^3.\end{aligned}$$

In (2.21), taking $b_1 = p/q$, and then multiplying both the sides by q^{12} we get

$$\begin{aligned}&2(4p(p+q))^6 + (4p^2 - 2pq - q^2)^6 = 2(4p^2+q^2)^6 \qquad (2.22)\\ &\quad + (16p^4 + 48p^3q + 12p^2q^2 + 4pq^3 - q^4)^3.\end{aligned}$$

Now, based on (2.22) and (2.20) we have the following two theorems:

Theorem 2.2. *The Diophantine equation* $2A^6 + B^6 = 2C^6 + D^3$ *has infinitely many nontrivial and primitive solutions in positive integers* $(A, B, C, D) = \{4p(p+q), (4p^2 - 2pq - q^2), (4p^2 + q^2), (16p^4 + 48p^3q + 12p^2q^2 + 4pq^3 - q^4)\}$ *where* $p, q \in \mathbb{N}$ *such that either (i).* $p = q = 1$*, or (ii).* $p > q$*,* $gcd(2p, q) = 1$*, and* $(p+q)$ *has prime factors* $\alpha_{i,\, i\in\mathbb{N}} \equiv 2, or\, 3(mod\, 4)$.

Proof. In (2.22), we have already established that

$$\begin{aligned}&2(4p(p+q))^6 + (4p^2 - 2pq - q^2)^6 = 2(4p^2+q^2)^6\\ &\quad + (16p^4 + 48p^3q + 12p^2q^2 + 4pq^3 - q^4)^3.\end{aligned}$$

When $p = q = 1$, we get $(A, B, C, D) = (8, 1, 5, 79)$ where $gcd(8, 1, 5, 79) = 1$. The conditions: $p, q \in \mathbb{N}$, and $p > q$, make (A, B, C, D) always positive for infinitely many (p, q) pairs. Since $gcd(2p, q) = 1$, q is odd; and $(4p^2+q^2)$ contains prime factors $\beta_{j,\, j\in\mathbb{N}} \equiv 1(mod\, 4)$. So, $gcd((4p^2 + q^2), (p+q)) = 1$; and $gcd((4p^2 + q^2), 4p) = 1$. Thus, we see that $gcd(A, C) = 1$, which implies that $gcd(A, B, C, D) = 1$. Thus, under the given conditions, we get infinitely many nontrivial and primitive solutions for (A, B, C, D). □

Example 2.3.

$$(p, q) = (2, 1): \qquad 2 \times 24^6 + 11^6 = 2 \times 17^6 + 695^3;$$
$$(p, q) = (4, 3): \qquad 2 \times 112^6 + 31^6 = 2 \times 73^6 + 15391^3.$$

Remark 2.4. *In Theorem 2.2, if we allow* $(p+q)$ *to have a prime factor* $\gamma \equiv 1(mod\,4)$*, then, there is no guarantee that* $gcd(A,B,C,D)$ *will always be 1 as one can easily verify from Table 2.1.*

Table 2.1

$\gamma = 5$,	$(p,q) = (4,1)$:	$2 \times 80^6 + 55^6 = 2 \times 65^6 + 7375^3$; $gcd(80,55,65,7375) = 5$.
$\gamma = 13$,	$(p,q) = (12,1)$:	$2 \times 624^6 + 551^6 = 2 \times 577^6 + 416495^3$; $gcd(624,551,577,416495) = 1$.

Theorem 2.5. *The Diophantine equation* $2A^6 + B^6 = 2C^6 - D^3$ *has infinitely many nontrivial and primitive solutions in positive integers*

$$(A,B,C,D) = \{4mn, (5m^2-n^2), (5m^2+2mn+n^2), (25m^4+40m^3n+6m^2n^2+8mn^3+n^4)\},$$

where $m,n \in \mathbb{N}$ *such that* $gcd(5m,n) = 1, 2m > n$*, and one is odd, the other is even.*

Proof. We show that

$$2(4mn)^6 + (5m^2 - n^2)^6 = (5m^2 + 2mn + n^2)^6 - (25m^4 + 40m^3n + 6m^2n^2 + 8mn^3 + n^4)^3, \quad (2.23)$$

by substituting $b_1 = m$, and $b_2 = n$ in (2.20). The conditions: $m,n \in \mathbb{N}$, and $2m > n$, make (A,B,C,D) always positive for infinitely many (m,n) pairs. The condition $gcd(5m,n) = 1$ tells that both of m and n are not even, and 5 is not a factor of n. Since both of m and n are not odd, $B = (5m^2 - n^2)$ is odd, and B does not share a common factor with $A = 4mn$. Thus, we prove that $gcd(A\,, B\,, C\,, D) = 1$, so that the numerical solutions we get are primitive. □

Example 2.6.

$$(m,n) = (2,1): \qquad 2 \times 8^6 + 19^6 = 2 \times 25^6 - 761^3;$$
$$(m,n) = (3,2): \qquad 2 \times 24^6 + 41^6 = 2 \times 61^6 - 4609^3.$$

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