Notes on Number Theory and Discrete Mathematics
ISSN 1310–5132
Vol. 20, 2014, No. 2, 10–19

Fibonacci primes of special forms

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Abstract: A study of Fibonacci primes of the form $x^2 + ry^2$ (where $r = 1$; $r =$ prime or $r =$ perfect power) is provided.
Keywords: Fibonacci numbers; quadratic fields; computational number theory; algebraic number theory computations.
AMS Classification: 11D25, 11S15, 11Y40, 11Y50.

1 Introduction

The prime numbers that can be written as x^2+ny^2, for $n\in\mathbb{N}^*$, have been studied in [3]. Necessary and sufficient conditions for a prime p to be written as $p = x^2 + ny^2$, with $n \in\mathbb{N}^*$, have been determined. One of the results of Cox's book [3] is:

Proposition 1.1. [3] *Let n be a square free positive integer which is not congruent to 3 modulo 4. Then there exists a monic irreducible polynomial $f\in\mathbb{Z}[X]$ of degree $h(\Delta)$, such that, if p is an odd prime that doesn't divide n or the discriminant of f and $E = HCF(K)$ is the Hilbert class field of $K = \mathbb{Q}(\sqrt{-n})$, the following statements are equivalent:*
(i) $p = x^2 + ny^2$, for some $x, y\in\mathbb{N}$.
(ii) p completely splits in E.
(iii) $\left(\frac{-n}{p}\right) = 1$ and the congruence $f(x) \equiv 0$ (mod p) has solutions in $\mathbb{Z}$.
Moreover, f is the minimal polynomial of a real algebraic integer α such that $E = K(\alpha)$.

In [5], [6] is given a characterization of some such primes p, when $n\equiv3$ (mod 4) and the class number of the quadratic field $\mathbb{Q}\left(\sqrt{-n}\right)$ is 1, namely $n\in\{11, 19, 43, 67, 163\}$: p is represented by $x^2 + ny^2$ if and only if the corresponding cubic field equation splits completely modulo p if and only if the roots of the resolvent quadratic equation are cubic residues of p. The field equations and the corresponding root α_n can be taken as:

n	field equation	root α_n of the resolvent
11	$x^3+6x-34=0$	$17+3\sqrt{33}$
19	$x^3-2x+2=0$	$27+3\sqrt{57}$
43	$x^3-4x-4=0$	$54+6\sqrt{129}$
67	$x^3-30x-106=0$	$53+3\sqrt{201}$
163	$x^3-8x-10=0$	$135+3\sqrt{489}$

Theorem 1.1. [6] *For* $q\in\{11,19,43,67,163\}$ *and for* α_q *defined above, a prime positive integer number* $p\equiv 1$ *(mod* 12*) such that the Legendre symbol* $\left(\frac{p}{q}\right)=1$ *is represented by* $p=x^2+qy^2$*, if and only if the cubic character* $\left(\frac{\alpha_q}{p}\right)_3=1$.

In this paper we try to determine the prime Fibonacci numbers that can be written in the form x^2+ry^2, where $r=1$, r is a prime natural number or r is a power of a prime number.
Recall that the Fibonacci sequence is defined by:

$$(F_n)_{n\geq 0}, F_0=0, F_1=1, F_{n+2}=F_{n+1}+F_n, n\geq 0.$$

Sometimes the sequence is given under the form:

$$F_n=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

In [2], Luca and Ballot studied the Fibonacci numbers $F_n=|x^2+dy^2|$, from another point of view. By denoting $N_d=\{n>0: F_n=|x^2+dy^2|, x,y\in\mathbb{Z}\}$, they proved that for any $d=\pm t^2$, $t\in\mathbb{N}$, the set N_d has positive lower asymptotic density.
We recall some properties of quadratic fields which are necessary in our proofs.

Proposition 1.2. [1] *Let* p,q *be two dictinct prime numbers,* $p\equiv q\equiv 1$ *(mod* 4*) and* h *the class number of the biquadratic field* $K=\mathbb{Q}\left(\sqrt{p},\sqrt{q}\right)$. *If* $\left(\frac{p}{q}\right)=1$, *then* h *is odd if and only if* $\left(\frac{p}{q}\right)_4\neq\left(\frac{q}{p}\right)_4$ *(here* $(\,)_4$ *is the quartic character).*

Proposition 1.3. [3] *Let* K *be an algebraic number field and* $P\in Spec(O_K)$*. Then* P *completely splits in the ring of integers of the Hilbert class field of* K *if and only if* P *is a principal ideal in the ring* O_K.

Proposition 1.4. [9]*Let* p *be a prime number. Then:*

(i) There exist integers x,y *such that* $p=x^2+y^2$ *if and only if* $p=2$ *or* $\left(\frac{-1}{p}\right)=1$;

(ii) There exist integers x,y *such that* $p=x^2+2y^2$ *if and only if* $\left(\frac{-2}{p}\right)=1$,

where ( ) denotes the Legendre symbol.

The following properties of Fibonacci numbers we will use in the following.

Proposition 1.5. [14] *The cycle of the Fibonacci numbers mod* 8 *is*

$$0,1,1,2,3,5,0,5,5,2,7,1,(0,1,1,2,3,5,0,5,5,2,7,1,),...$$

so the cycle-length of the Fibonacci numbers mod 8 *is* 12*.*

Proposition 1.6. [14] *Let* $(F_n)_{n\geq 0}$ *be the Fibonacci sequence. If* F_n *is a prime number,* $n \geq 5$*, then* n *is a prime number.*

Proposition 1.7. [14] *A Fibonacci number* F_n *is even if and only if* $n \equiv 0$ *(mod* 3*).*

Theorem 1.2. [11] ***(Legendre, Lagrange)*** *If* p *is an odd prime number, then*

$$F_p \equiv \left(\frac{p}{5}\right)(modp).$$

Theorem 1.3. [11] ***(Legendre, Lagrange)*** *Let* p *be an odd prime number. Then*

$$F_{p-1} \equiv \frac{1-\left(\frac{p}{5}\right)}{2}(modp)$$

and

$$F_{p+1} \equiv \frac{1+\left(\frac{p}{5}\right)}{2}(modp).$$

Theorem 1.4. [11] , [12] *Let* $p \notin \{2,5\}$ *be an odd prime number. Then*

$$F_{\frac{p-\left(\frac{p}{5}\right)}{2}} \equiv \begin{cases} 0\,(mod\,p), if\ p \equiv 1(mod4), \\ 2\,(-1)^{\left[\frac{p+5}{10}\right]} \cdot \left(\frac{p}{5}\right) \cdot 5^{\frac{p-3}{4}}\ (mod\,p),\ if\ \ p \equiv 3(mod4) \end{cases}$$

where $[x]$ *is the integer part of x.*

Proposition 1.8. [7] *Let* K *be an algebriac numbers field and* h_K *the class number of* K *and let* p *be a prime positive integer,* p *does not divide* h_K*. Let* I *be a nonzero integer ideal in the ring of integers of* K *such that* I^p *is principal. Then* I *is principal.*

2 Fibonacci primes of the form $x^2 + ry^2$

Our first remark is:

Remark 2.1. *i) If* p *is a prime Fibonacci number,* $p \neq 3$*, then there exist the integers* x, y *such that* $p = x^2 + y^2$*.*

ii) If p *is a prime Fibonacci number,* $p \equiv 1$ *(mod* 8*), then there exist the integers* x, y *such that* $p = x^2 + 2y^2$*.*

Proof. i) **Case 1.** $p = 2 = F_3$. We obtain $2 = 1^2 + 1^2$.

Case 2. $p = F_m \geq 5$ is an odd prime number, applying Proposition 1.6 it results that m is an odd prime number.

The assertion results from the identity: $F_{2n+1} = F_n^2 + F_{n+1}^2, g.c.d\,(F_n, F_{n+1}) = 1$.

Thus, all the odd prime Fibonacci numbers $p = F_n$ are congruent with 1 (mod 4) and are sums of two perfect squares.

ii) If $p \equiv 1$ (mod 8) is a Fibonacci prime number (see Proposition 1.5), applying Proposition 1.4 (ii) we obtain that there exist two integers x, y such that $p = x^2 + 2y^2$.

A natural idea is to ask ourselves if there exist Fibonacci numbers F_p of the form $F_p = x^2 + p^2y^2$, where p is a prime positive integer. The following result has been obtained:

Proposition 2.1. *i) For each* p*, a prime number,* $p \geq 7$*,* $p \equiv 1$ *(mod* 4*), there exist integer numbers* x, y *so that, the Fibonacci number* F_p *can be written as* $F_p = x^2 + p^2y^2$.

ii) For each p*, a prime numberr,* $p \geq 7$*,* $p \equiv 1$ *(mod* 4*), with the property that the Fibonacci number* F_p *is a prime number, there exist a unique pair of positive integer numbers* x, y *so that the Fibonacci number* F_p *can be written as:* $F_p = x^2 + p^2y^2$.

Proof. i) It is known that $F_{2n+1} = F_n^2 + F_{n+1}^2$, so, if p is an odd prime number, then $F_p = F_{\frac{p+1}{2}}^2 + F_{\frac{p-1}{2}}^2$. Since the Legendre symbol $\left(\frac{p}{5}\right)$ is 1 when $p \equiv 1, 4$ (mod 5) and it is -1 when $p \equiv 2, 3$ (mod 5), we divide the proof in two cases: 1: $p \equiv 1, 4$ (mod 5); 2. $p \equiv 2, 3$ (mod 5).

Case 1: $p \equiv 1, 4$ (mod 5).

Using the fact that $p \equiv 1$ (mod 4) and Chinese Remainder Theorem it results that $p \equiv 1, 9$ (mod 20). Applying Theorem 1.4, it results that $F_{\frac{p-1}{2}} \equiv 0$ (mod p). Therefore, there exist integer numbers x, y, $x = \pm F_{\frac{p+1}{2}}$ and $y = \pm \frac{F_{\frac{p-1}{2}}}{p}$ such that $F_p = x^2 + p^2y^2$.

Case 2: $p \equiv 2, 3$ (mod 5).

From $p \equiv 1$ (mod 4) and Chinese Remainder Theorem it results that $p \equiv 13, 17$ (mod 20). Applying Theorem 1.4, it results that $F_{\frac{p+1}{2}} \equiv 0$ (mod p). We obtain that there exist integer numbers x, y, $x = \pm F_{\frac{p-1}{2}}$ and $y = \pm \frac{F_{\frac{p+1}{2}}}{p}$ such that $F_p = x^2 + p^2y^2$.

ii) If moreover, the Fibonacci number F_p is a prime number, applying (i) and the properties that $\mathbb{Z}[i]$ is a factorial ring and its group of units is $U(\mathbb{Z}[i]) = \{1, -1, i, -i\}$ and $F_p \equiv 1$ (mod 4) completely splits in the ring $\mathbb{Z}[i]$, it results that there exist unique positive integer numbers x, y, $x = F_{\frac{p+1}{2}}$ and $y = \frac{F_{\frac{p-1}{2}}}{p}$, when $p \equiv 1, 9$ (mod 20), respectively $x = F_{\frac{p-1}{2}}$ and $y = \frac{F_{\frac{p+1}{2}}}{p}$ when $p \equiv 13, 17$ (mod 20), such that $F_p = x^2 + p^2y^2$.

Remark 2.2. *i) The condition* $p \equiv 1$ *(mod* 4*) is necessary in the statement of Proposition 2.1. Otherwise, if* $p = 7 \equiv 3$ *(mod* 4*), we have* $F_7 = F_3^2 + F_4^2 = 2^2 + 3^2$*, but* 7 *does not divide* F_3*,* 7 *does not divide* F_4*. If* $p = 11 \equiv 3$ *(mod* 4*), we have* $F_{11} = F_5^2 + F_6^2 = 5^2 + 8^2$*, but* 11 *does not divide* F_5*,* 11 *does not divide* F_6*.*

ii) The decomposition of a non - prime positive integer, congruent with 1 *mod* 4 *as a sum of two square is not unique. For example:* $F_{19} = 4181 = 34^2 + 55^2 = 41^2 + 50^2$.

Proposition 2.2. *For each positive integer* n*,* $n \equiv 7$ *(mod* 16*), there exist integer numbers* x, y *so that, the Fibonacci number* F_n *can be written as* $F_n = x^2 + 3^2y^2$.

Proof. Using again that $F_{2n+1} = F_n^2 + F_{n+1}^2$ and also that $F_{2n} = F_n \cdot L_n$ so, if n is an odd positive integer, $n \equiv 7$ (mod 16) then $F_n = F_{\frac{n-1}{2}}^2 + F_{\frac{n+1}{2}}^2 = F_{\frac{n-1}{2}}^2 + F_{\frac{n+1}{4}}^2 \cdot L_{\frac{n+1}{4}}^2$.
We remark that $\frac{n+1}{4} \equiv 2$ (mod 4) and we can prove imediately that $L_{\frac{n+1}{4}} \equiv 0$ (mod 3). Now, the conclusion of the Proposition is proved.

In the following we study the Fibonacci primes F_p of the form $x^2 + py^2$.

A first example of a such a prime is $5 = F_5 = 0^2 + 5 \cdot (\pm 1)^2$.

We wish to determine the primes F_p of the form $x^2 + py^2$, with x, y positive integers. With a simple computation in MAGMA sofware ([15]), we obtain:

$R\langle x\rangle$:=PolynomialRing(Integers());
$f := x^2 + 29$;
T:=Thue(f);
T;
Solutions(T,514229);
Submit
Thue object with form: $X^2 + 29Y^2$
$[-552, 85]$,
$[552, -85]$,
$[552, 85]$,
$[-552, -85]$.
So,

$$514229 = F_{29} = (\pm 552)^2 + 29 \cdot (\pm 85)^2 .$$

Similarly:

$$233 = F_{13} = (\pm 5)^2 + 13 \cdot (\pm 4)^2 , 1597 = F_{17} = (\pm 38)^2 + 17 \cdot (\pm 3)^2 .$$

We remark that in all these examples $p \equiv 1$ (mod 4). Therefore, the question that arises is: what happens when $p \equiv 3$ (mod 4)? First, we tried to apply Theorem 1.1 for $p \in \{11, 19, 43, 67, 163\}$, but this was not possible because, using [11] we have: F_{11} is not congruent with 1 (mod 12), F_{19} is not a prime number, F_{43} is not congruent with 1 (mod 12), F_{67} and F_{163} are not prime numbers.

The following result holds true:

Proposition 2.3. *If p is a prime number, $p \equiv 3$ or 7 (mod 20) then there exists no Fibonacci number F_p of the form $x^2 + py^2$.*

Proof. Let p be a prime number, $p \equiv 3$ or 7 (mod 20). We suppose by reductio ad absurdum that there exists a Fibonacci number, F_p, such that $F_p = x^2 + py^2$. Therefore, the Legendre symbol $\left(\frac{F_p}{p}\right) = 1$. But, appplying Theorem 1.2 and the properties of Legendre' symbol, we have:

$$\left(\frac{F_p}{p}\right) = \left(\frac{\left(\frac{p}{5}\right)}{p}\right) = \begin{cases} \left(\frac{1}{p}\right) = 1, \text{if } p \equiv 1 \text{ or } 4 \ (mod\ 5), \\ \left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}, \text{ if } p \equiv 2 \text{ or } 3 \ (mod\ 5). \end{cases}$$

Since $p \equiv 3$ or $7 \pmod{20}$ it results that $p \equiv 2$ or $3 \pmod 5$. So,

$$\left(\frac{F_p}{p}\right) = \left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}} = -1.$$

We have obtained a contradiction with the fact that $\left(\frac{F_p}{p}\right) = 1$.

It is hard to know what happens when $p \equiv 11$ or $19 \pmod{20}$. With a simple computation in MAGMA sofware, we obtain that F_{131} can not be written in the form $x^2 + 131y^2$.

Similarly: F_{359} can not be written in the form $x^2 + 359y^2$, but F_{2971} can be written in the form $x^2 + 2971y^2$.

Now, we study the prime Fibonacci numbers F_p of the form $x^2 + py^2$, with $p \equiv 1 \pmod 4$.

First, we give the following remark:

Remark 2.3. *Let K be the biquadratic field $K = \mathbb{Q}\left(\sqrt{p}, \sqrt{F_p}\right)$ and let O_K be the ring of integers of this field. Then, the ideal $\left(\sqrt{F_p} + y\sqrt{p}\right) O_K$ is the square of an ideal of O_K.*

Proof. It is known that O_K is a Dedekind ring for every value of p and F_p. Consider the Diophantine equation $F_p = x^2 + py^2$. Passing to ideals, this Diophantine equation becomes

$$\left(\sqrt{F_p} - y\sqrt{p}\right) O_K \cdot \left(\sqrt{F_p} + y\sqrt{p}\right) O_K = x^2 O_K.$$

It is easy to show that the ideals $\left(\sqrt{F_p} - y\sqrt{p}\right) O_K$ and $\left(\sqrt{F_p} + y\sqrt{p}\right) O_K$ are coprimes. Looking to the last form of our equation and applying a property of Dedekind rings, it results that there exists an ideal J in the ring O_K such that

$$\left(\sqrt{F_p} + y\sqrt{p}\right) O_K = J^2 \tag{2.1}$$

Next, we make a few observations about the last result obtained.

Since $p \equiv F_p \equiv 1 \pmod 4$ and $\left(\frac{p}{F_p}\right) = 1$, it is known [7] that

$$\left(\frac{p}{F_p}\right)_4 = \left(\frac{F_p}{p}\right)_4 \cdot \left(\frac{\epsilon_p}{F_p}\right),$$

where ϵ_p is a fundamental unity of the field $\mathbb{Q}(\sqrt{p})$.

If $\left(\frac{\epsilon_p}{F_p}\right) = -1$, then $\left(\frac{p}{F_p}\right)_4 \neq \left(\frac{F_p}{p}\right)_4$, and applying Proposition 1.2, it results that h_K is odd. Applying Proposition 1.8 we obtain that J is a principal ideal in the ring O_K.

From the relation (2.1) it results that

$$\sqrt{F_p} + y\sqrt{p} = u\left(a + b\sqrt{F_p} + c\sqrt{p} + d\sqrt{pF_p}\right)^2, \tag{2.2}$$

where u is a unity in the ring O_K.

The fundamental system of unities of the ring O_K is $\left\{\epsilon_p, \epsilon_{F_p}, \sqrt{\epsilon_{pF_p}}\right\}$, when $N\left(\epsilon_{pF_p}\right) = -1$, respectively $\left\{\epsilon_p, \epsilon_{F_p}, \sqrt{\epsilon_p \epsilon_{F_p} \epsilon_{pF_p}}\right\}$, when $N\left(\epsilon_{pF_p}\right) = 1$ ([4], [8]).

Concluding, a characterization of prime Fibonacci numbers of the form $F_p = x^2 + py^2$, with $p \equiv 1 \pmod 4$ has been obtained. But, even if F_p is a prime number with special properties, it is hard to determine the solutions of (2.2). If h_K is an even number, it is harder to determine the solutions of equation (2.2).

In the following we try to give another characterization of prime Fibonacci numbers of the form $F_p = x^2 + py^2$, when $p \equiv 1 \pmod 4$, so $p \equiv 1$ or $5 \pmod{12}$ using techniques of computational number theory.

A natural question is: How many prime Fibonacci numbers of the form $F_p = x^2 + py^2$ do exist? From Proposition 1.1 it results that, when p is not congruent with $3 \pmod 4$, a prime Fibonacci number has the form $F_p = x^2 + py^2$ if and only if F_p completely splits in the ring of integers of the Hilbert class field for the quadratic field $\mathbb{Q}\left(\sqrt{-p}\right)$.

If we denote by $L = \mathbb{Q}\left(\sqrt{-p}\right)$, P_L - the set of all finite primes of L, $HCF(L)$ - the Hilbert class field of L, δ - the Cebotarev density, S - the set of prime from $\mathbb{N}$ which completely split in $HCF(L)$, and T - the set of prime Fibonacci numbers F_p of the form $F_p = x^2 + py^2$, applying a result of [3], the theorem of tranzitivity of finite extensions and Proposition 1.1, we obtain that: $T \subset S$ and

$$\delta\left(S\right) = \frac{1}{[L:\mathbb{Q}]\cdot[HCF(L):L]} = \frac{1}{2\cdot[HCF(L):L]} = \frac{1}{2\cdot h_L},$$

where h_L is the order of ideal class group of the ring of integers of L.

So $\delta\left(T\right) \leq \delta\left(S\right)$.

With a simple computation with MAGMA we obtain:

$Q :=$ Rationals();
$Z :=$ RingOfIntegers(Q);
Z;
$L :=$ QuadraticField(-50833);
L;
$O_L :=$ RingOfIntegers(L);
O_L;
ClassNumber(O_L);
Evaluate
Integer Ring
Quadratic Field with defining polynomial $.1^2 + 50833$ over the Rational Field
Maximal Equation Order of L
128.
So, for $L = \mathbb{Q}\left(\sqrt{-50833}\right)$, $h_L = 128$.

If we consider all prime Fibonacci numbers F_p, with $p \equiv 1$ or $5 \pmod{12}$ known up to now [12] and we calculate the class number for the field $L = \mathbb{Q}(\sqrt{-p})$, using MAGMA, we obtain:

$$h_{\mathbb{Q}\left(\sqrt{-13}\right)} = 2, h_{\mathbb{Q}\left(\sqrt{-17}\right)} = 4, h_{\mathbb{Q}\left(\sqrt{-29}\right)} = 6, h_{\mathbb{Q}\left(\sqrt{-137}\right)} = 8, h_{\mathbb{Q}\left(\sqrt{-449}\right)} = 20, h_{\mathbb{Q}\left(\sqrt{-509}\right)} = 30$$

$$h_{\mathbb{Q}\left(\sqrt{-569}\right)} = 32, h_{\mathbb{Q}\left(\sqrt{-9677}\right)} = 98, h_{\mathbb{Q}\left(\sqrt{-25561}\right)} = 88, h_{\mathbb{Q}\left(\sqrt{-30757}\right)} = 90, h_{\mathbb{Q}\left(\sqrt{-50833}\right)} = 128.$$

We remark that, when a prime p, $p \equiv 1$ or 5 (mod 12) increases, then $\delta(S)$ decreases. Using the procedure (in MAGMA) described after Proposition 2.2 or the procedure described below, and applying Propositions 1.3 and 1.1, it results that the only prime Fibonacci numbers F_p of the form $F_p = x^2 + py^2$, with $p < 10^4$ are $F_{13}, F_{17}, F_{29}, F_{2971}, F_{9311}, F_{9677}$.

$\mathbb{Q}$:= Rationals();
$\mathbb{Z}$:= RingOfIntegers($\mathbb{Q}$);
Z;
$\mathbb{Q} < t >$:= PolynomialRing($\mathbb{Q}$);
$f := t^2 + 17$;
$K < a >$:= NumberField(f);
a;
O := RingOfIntegers(K);
O;
P:=ideal $< \mathbb{Z}|1597 >$;
P;
IsPrime(P);
Decomposition($O, 1597$);
M := ideal $< O|1597, a + 545 >$;
IsPrime(M);
IsPrincipal(M);
Evaluate
-17
Maximal Equation Order with defining polynomial $x^2 + 17$ over $\mathbb{Z}$
Ideal of Integer Ring generated by 1597
true
[
<Prime Ideal of O
Two element generators:
$[1597, 0]$
$[545, 1], 1 >$
<Prime Ideal of O
Two element generators:
$[1597, 0]$
$[1052, 1], 1 >$
]
Ideal of O
Two element generators:
$[1597, 0]$
$[545, 1]$
true
true

Using Proposition 1.1 we obtain:

Corollary 2.1. *All Fibonacci primes F_p, with $p < 10^4$, p is not congruent with 3 (mod 4) which splits completely in the ring of integers of the Hilbert class field for the quadratic field $L = \mathbb{Q}(\sqrt{-p})$ are $F_{13}, F_{17}, F_{29}, F_{9677}$.*

3 Conclusions

In this paper we have obtained certain characterizations of Fibonacci numbers of the form $F_p = x^2 + py^2$, with x, y integer numbers. We proved that there are no prime Fibonacci numbers of this form when $p \equiv 3, 7$ (mod 20). We think that there are no Fibonacci primes of this form, when $p \equiv 11, 19$ (mod 20) and we intend to study this problem in the future. We also gave elementary, combinatorial and algebraic characterizations for the studied numbers.

Acknowledgements

The author is very grateful to A. Mouhib for helpful discussions about this paper. Since the author had the first ideas of this paper during her visit to the International Centre for Theoretical Physics (ICTP) in Trieste, she wishes to express her thanks to Department of Mathematics of ICTP for the hospitality and support. Also, the author thanks the referees for helpful comments and suggestions.

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