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A set of Lucas sequences

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Abstract: A new extension of the concept of Fibonacci–like sequences is constructed, related to Lucas sequence. Some of its properties are discussed.
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23 March 2014. My 60-th anniversary.

1 Introduction

During the last century, the Fibonacci sequence has been extended in different ways by a variety of authors [1 – 9]. Ten years ago, Vassia Atanassova, Tony Shannon and the author introduced a new type of modification of the Fibonacci sequence [9].

Let the real numbers a, b, c be given. Then we can construct the following set of sequences:

$$
\begin{array}{llllll}
 & & & & 2a+3b & ... \\
 & & & & 2a+2b+c & ... \\
 & & & a+2b & 2a+2b+c & ... \\
 & b & a+b & a+b+c & 2a+b+2c & ... \\
a & & & & & \\
 & c & a+c & a+b+c & 2a+2b+c & ... \\
 & & & a+2c & 2a+b+2c & ... \\
 & & & & 2a+b+2c & ... \\
 & & & & 2a+3c & ...
\end{array}
$$

Here we introduce a modification of this modification, but directed to Lucas sequence.

2 Definition of a set of extensions of Lucas sequence

Let the real numbers a, b, c, d, e, f be given. Then we can construct the following set of sequences:

$$
\begin{array}{lllll}
 & & & a+2e & \dots \\
 & & & a+e+f & \dots \\
 & & & b+2e & \dots \\
 & & & b+e+f & \dots \\
 & & a+e & c+2e & \dots \\
 & & a+f & c+e+f & \dots \\
a & & d+e & d+2e & \dots \\
b & e & b+f & d+e+f & \dots \\
c & f & c+e & a+e+f & \dots \\
d & & c+f & a+2f & \dots \\
 & & d+e & b+e+f & \dots \\
 & & d+f & b+2f & \dots \\
 & & & c+e+f & \dots \\
 & & & c+2f & \dots \\
 & & & d+e+f & \dots \\
 & & & d+2f & \dots
\end{array}
$$

If we take one member from each column, we obtain separate sequences, each one of which is of Fibonacci (and Lucas) type. For example,

$$\{a, e, a+e, a+2e, ...\},$$

$$\{b, f, c+e, d+2f, ...\}.$$

3 Properties

It can be seen directly that if C_i is the number of members of the i-th column ($i = 0, 1, 2, ...$), then these numbers can be expressed in terms of elements of the Lucas sequence

$$\{L_n\}_{n=0}^{\infty} = \{2, 1, 3, 4, ...\}:$$

$$C_0 = 4 = 2^{L_0},\ C_1 = 2 = 2^{L_1},\ C_2 = 8 = 2^{L_2},\ C_3 = 16 = 2^{L_3}, ...$$

This is the reason for the name of the new object – set of extensions of the Lucas sequence.

It can be proved, e.g., by mathematical induction, that in the i-th column ($i = 0, 1, 2, ...$) there are

$$C_i = 2^{L_i}$$

members.

Let $\varphi_{i,j}$ be the member of one of the new sequences located in the j-th row of the i-th column: $i = 0, 1, 2, ...; 1 \le j \le 2^{L_i}$. Among the results which can be established is that if we have two consecutive elements $\varphi_{i,j}$ and $\varphi_{i+1,k}$ of one of the new sequences, then we can determine the next element of the sequence. It has the form:

$$\varphi_{i+2,2^{L_{i+1}}.(j-1)+k} = \varphi_{i,j} + \varphi_{i+1,k}. \tag{1}$$

For instance, when $i = 1, j = 2, k = 4$, we have

$$\begin{aligned}
\varphi_{i,j} + \varphi_{i+1,k} &= \varphi_{1,2} + \varphi_{2,4} \\
&= f + (b + f) \\
&= 2f + b \\
&= \varphi_{3,12} \\
&= \varphi_{i+2,2^3+k} \\
&= \varphi_{i+2,2^{L_2+1}.(j-1)+k}.
\end{aligned}$$

On the other hand, if the element $\varphi_{i,s}$ has been given for some natural numbers i, s, then, for fixed i, we can solve the linear Diophantine equation

$$s = 2^{L_{i+1}}.(j-1) + k, \tag{2}$$

for j, k, with the restrictions $1 \leq j;\ k \leq 2^{L_{i+1}}$. Solutions for (2) include

$$j = \left[\frac{s}{2^{L_{i+1}}}\right] + 1, \tag{3}$$

$$k = s - 2^{L_{i+1}}.\left[\frac{s}{2^{L_{i+1}}}\right]. \tag{4}$$

For example, when $i = 2, s = 4$, we get $j = 1, k = 4$, and equation (1) becomes

$$\begin{aligned}
\varphi_{4,4} &= \varphi_{2,1} + \varphi_{3,4} \\
&= e + b + f \\
&= \varphi_{2,1} + \varphi_{3,4}.
\end{aligned}$$

Thus, for example, when $a = 2, b = 0, c = 0, d = 0, e = 1, f = 0$, we obtain the set of Lucas sequences.

			4	...
			3	...
			2	...
			1	...
		3	2	...
		2	1	...
2		1	2	...
0	1	0	1	...
0	0	1	3	...
0		0	2	...
		1	1	...
		0	0	...
			1	...
			0	...
			1	...
			0	...

This set contains the Lucas sequence, as an element. If we like to obtain only Lucas sequences, we must put $a = b = c = d = 2, e = f = 1$.

It is easily seen that the two sets from [9] are subsets of the above set. Indeed, the sets from [9] have the forms:

$$
\begin{array}{llllll}
 & & & & 2p+3q & \dots \\
 & & & & 2p+2q+r & \dots \\
 & & & p+2q & 2p+2q+r & \dots \\
 & q & p+q & p+q+r & 2p+q+2r & \dots \\
p & & & & & \\
 & r & p+r & p+q+r & 2p+2q+r & \dots \\
 & & & p+2r & 2p+q+2r & \dots \\
 & & & & 2p+q+2r & \dots \\
 & & & & 2p+3r & \dots
\end{array}
\qquad
\begin{array}{lllll}
 & & & p+2q & \dots \\
 & & & p+q+s & \dots \\
 & & p+q & 2q+r & \dots \\
p & q & p+s & q+r+s & \dots \\
r & s & q+r & p+q+s & \dots \\
 & & r+s & p+2s & \dots \\
 & & & q+r+s & \dots \\
 & & & r+2s & \dots
\end{array}
$$

Then, for the first set we put

$$a = p, b = c = d = 0, e = q, f = r,$$

for the second one

$$a = p, b = r, c = d = 0, e = q, f = s$$

and obtain, respectively, the Lucas sets

$$
\begin{array}{lllll}
 & & & p+2q & \dots \\
 & & & p+q+r & \dots \\
 & & & 2q & \dots \\
 & & & q+r & \dots \\
 & & p+q & 2q & \dots \\
 & & p+r & q+r & \dots \\
p & & q & 2q & \dots \\
0 & q & r & q+r & \dots \\
0 & r & q & p+q+r & \dots \\
0 & & r & p+2r & \dots \\
 & & q & q+r & \dots \\
 & & r & 2r & \dots \\
 & & & q+r & \dots \\
 & & & 2r & \dots \\
 & & & q+r & \dots \\
 & & & 2r & \dots
\end{array}
\qquad
\begin{array}{lllll}
 & & & p+2q & \dots \\
 & & & p+q+r & \dots \\
 & & & 2q & \dots \\
 & & & q+r & \dots \\
 & & p+q & 2q & \dots \\
 & & p+r & q+r & \dots \\
p & & q & 2q & \dots \\
0 & q & r & q+r & \dots \\
0 & r & q & p+q+r & \dots \\
0 & & r & p+2r & \dots \\
 & & q & q+r & \dots \\
 & & r & 2r & \dots \\
 & & & q+r & \dots \\
 & & & 2r & \dots \\
 & & & q+r & \dots \\
 & & & 2r & \dots
\end{array}
$$

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