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n-Pulsated Fibonacci sequence

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Abstract: A new type of Fibonacci sequence is introduced and explicit formulas for the form of its members are formulated and proved. It is an extension of the special Fibonacci sequence, introduced in [1] and called a Pulsated Fibonacci sequence.
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To my colleague and friend Prof. Ivan Dotsinsky
for his 80th birthday!

1 Introduction

In the last year 2013, two new modifications of the Fibonacci sequence were introduced in [1, 2]. Here, in continuation of this direction of research, an extension of the new type of Fibonacci–like sequence is introduced.

First, we give the two modifiations from [1, 2].

Let a and b be two fixed real numbers. The following two sequences

$$\alpha_0 = a,\ \beta_0 = b$$

$$\alpha_{2k+1} = \beta_{2k+1} = \alpha_{2k} + \beta_{2k},$$

$$\alpha_{2k+2} = \alpha_{2k+1} + \beta_{2k},$$

$$\beta_{2k+2} = \beta_{2k+1} + \alpha_{2k},$$

for the natural numner $k \geq 0$ are called (a, b)-*Pulsated Fibonacci sequence* (see [1]).

Let a, b and c be three fixed real numbers. The following two sequences

$$\alpha_0 = a,\ \beta_0 = b$$

$$\alpha_1 = \beta_1 = c$$

$$\alpha_{2k} = \alpha_{2k-1} + \beta_{2k-2},$$

$$\beta_{2k} = \beta_{2k-1} + \alpha_{2k-2},$$

$$\alpha_{2k+1} = \beta_{2k+1} = \alpha_{2k} + \beta_{2k},$$

for the natural number $k \geq 1$ are called $(a, b; c)$-*Pulsated Fibonacci sequence* (see, [2]).

Now, we extend both sequences.

2 First extension

Let $a_1, a_2, ..., a_n$ be n fixed real numbers. The following n sequences

$$\alpha_{1,0} = a_1,\ \alpha_{2,0} = a_2,\ ...\ \alpha_{n,0} = a_n,$$

$$\alpha_{1,2k+1} = \alpha_{2,2k+1} = ... = \alpha_{n,2k+1} = \alpha_{1,2k} + \alpha_{2,2k} + ... = \alpha_{n,2k},$$

$$\alpha_{1,2k+2} = \alpha_{1,2k+1} + \alpha_{n,2k},$$

$$\alpha_{2,2k+2} = \alpha_{2,2k+1} + \alpha_{n-1,2k},$$

$$\cdot\ \cdot\ \cdot$$

$$\alpha_{n,2k+2} = \alpha_{n,2k+1} + \alpha_{1,2k+2},$$

for the natural number $k \geq 0$ are called $(a_1, a_2, ..., a_n)$-*Pulsated Fibonacci sequence.*

Let

$$A = \sum_{i=1}^{n} a_i. \tag{1}$$

The first values of the new sequence are given in the following Table 1.

Table 1.

k	$\alpha_{1,k}$	$\alpha_{2,k}$	...	$\alpha_{n,k}$
0	a_1	a_2	...	a_n
1	A	A	...	A
2	$A + a_n$	$A + a_{n-1}$	...	$A + a_1$
3	$(n+1)A$	$(n+1)A$	...	$(n+1)A$
4	$(n+2)A + a_1$	$(n+2)A + a_2$	...	$(n+2)A + a_n$
5	$(n+1)^2A$	$(n+1)^2A$	...	$(n+1)^2A$
6	$(n^2+3n+3)A + a_1$	$(n^2+3n+3)A + a_2$	...	$(n^2+3n+3)A + a_n$
7	$(n+1)^3A$	$(n+1)^3A$	...	$(n+1)^3A$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Theorem 1. For every two natural numbers $k \geq 0$ and $i, 1 \leq i \leq n$,

$$\alpha_{i,4k+2} = \frac{(n+1)^{2k+1} - 1}{n} A + a_{n+1-i} \tag{2}$$

$$\alpha_{i,4k+3} = (n+1)^{2k+1} A, \tag{3}$$

$$\alpha_{i,4k+4} = \frac{(n+1)^{2k+2} - 1}{n} A + a_i \tag{4}$$

$$\alpha_{i,4k+5} = (n+1)^{2k+2} A. \tag{5}$$

Proof. Let below i be a fixed number, so that $1 \leq i \leq n$.

Obviously, for $k = 0$ the assertion is valid. Let us assume that for some natural number $k \geq 0$, (2) – (5) are valid. For the natural number $k+1$, first, we check that

$$\alpha_{i,4k+6} = \alpha_{i,4k+5} + \alpha_{n+1-i,4k+4}$$

$$= (n+1)^{2k+2} A + \frac{(n+1)^{2k+2} - 1}{n} A + a_{n+1-i}$$

$$= \frac{n(n+1)^{2k+2} + (n+1)^{2k+2} - 1}{n} A + a_{n+1-i}$$

$$= \frac{(n+1)^{2k+3} - 1}{n} A + a_{n+1-i}.$$

Therefore, (2) is valid for $k+1$.

Second, we check that

$$\alpha_{i,4k+7} = \sum_{j=1}^{n} \alpha_{j,4k+6}$$

$$= \sum_{j=1}^{n} \left(\frac{(n+1)^{2k+3} - 1}{n} A + a_{n+1-j} \right)$$

$$= ((n+1)^{2k+3} - 1) A + \sum_{j=1}^{n} a_{n+1-j}$$

$$= ((n+1)^{2k+3} - 1) A + A = (n+1)^{2k+3} A.$$

Therefore, (3) is valid for $k+1$. The two other checks are similar. □

Obviously, when $n = 2, a_1 = a, a_2 = b$, we obtain the (a, b)-Pulsated Fibonacci sequence discussed above.

3 Second extension

Let $a_1, a_2, ..., a_n$ and c be $n+1$ fixed real numbers. The following n sequences

$$\alpha_{1,0} = a_1,\ \alpha_{2,0} = a_2,\ ...\ \alpha_{n,0} = a_n,$$

$$\alpha_{1,1} = \alpha_{2,1} =\ ...\ \alpha_{n,1} = c,$$

$$\alpha_{1,2k+2} = \alpha_{1,2k+1} + \alpha_{n,2k},$$

$$\alpha_{2,2k+2} = \alpha_{2,2k+1} + \alpha_{n-1,2k},$$

$$\cdot\ \cdot\ \cdot$$

$$\alpha_{n,2k+2} = \alpha_{n,2k+1} + \alpha_{1,2k},$$

$$\alpha_{1,2k+1} = \alpha_{2,2k+1} = ... = \alpha_{n,2k+1} = \alpha_{1,2k} + \alpha_{2,2k} + ... = \alpha_{n,2k},$$

for the natural number $k \geq 0$ are called $(a_1, a_2, ..., a_n; c)$-*Pulsated Fibonacci sequence.*

Let A again satisfy (1).

The first values of the new sequence are given in the following Table 2.

Table 2.

k	$\alpha_{1,k}$	$\alpha_{2,k}$	...	$\alpha_{n,k}$
0	a_1	a_2	...	a_n
1	c	c	...	c
2	$a_n + c$	$a_{n-1} + c$	...	$a_1 + c$
3	$A + nc$	$A + nc$	...	$A + nc$
4	$a_1 + A + (n+1)c$	$a_2 + A + (n+1)c$	...	$a_n + A + (n+1)c$
5	$(n+1)A + n(n+1)c$	$(n+1)A + n(n+1)c$	...	$(n+1)A + n(n+1)c$
6	$a_n+(n+2)A+(n+1)^2c$	$a_{n-1}+(n+2)A+(n+1)^2c$	...	$a_1+(n+2)A+(n+1)^2c$
7	$(n+1)^2A + n(n+1)^2c$	$(n+1)^2A + n(n+1)^2c$	...	$(n+1)^2A + n(n+1)^2c$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Theorem 2. For every two natural numbers $k \geq 0$ and $i, 1 \leq i \leq n$,

$$\alpha_{i,4k+2} = a_{n+1-i} + \frac{(n+1)^{2k} - 1}{n}A + (n+1)^{2k}c,$$

$$\alpha_{i,4k+3} = (n+1)^{2k}A + n(n+1)^{2k}c,$$

$$\alpha_{i,4k+4} = a_i + \frac{(n+1)^{2k+1} - 1}{n}A + (n+1)^{2k+1}c,$$

$$\alpha_{i,4k+5} = (n+1)^{2k+1}A + n(n+1)^{2k+1}c.$$

The proof follows by analogy.

References

[1] Atanassov, K., Pulsating Fibonacci sequences. *Notes on Number Theory and Discrete Mathematics*, Vol. 19, 2013, No. 3, 12–14.

[2] Atanassov, K., Pulsated Fibonacci sequence. Part 2, *Notes on Number Theory and Discrete Mathematics*, Vol. 19, 2013, No. 4, 33–36.