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The decimal string of the golden ratio

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Abstract: The decimal expansion of the Golden Ratio is examined through the use of various properties of the Fibonacci numbers and some exponential functions.
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1 Introduction

It is well-known that the powers of the Golden Ratio, Phi or φ, are related to the elements of the Fibonacci sequence, $\{F_n\}$, [1,6]:

$$\varphi^n = F_n\varphi + F_{n-1}, \tag{1.1}$$

and

$$\psi^n = F_{n+1} - F_n\phi, \tag{1.2}$$

in which

$$\psi = 1 - \varphi.$$

Similarly for the Lucas sequence, $\{L_n\}$, [7]:

$$\psi^n\Psi = L_{n-1}\psi + L_{n-2}, \tag{1.3}$$

and

$$\varphi^n\Phi = L_n - L_{n-1}\psi. \tag{1.4}$$

in which

$$\Psi = \psi + 2$$

and

$$\Phi = \varphi + 2$$

Are the roots of $0 = x^2 - 3x + 1$, which is the characteristic polynomial of the second order homogeneous linear recurrence relation

$$U_n = 3U_{n-1} - U_{n-2},$$

from which even- and odd-suffixed Fibonacci and Lucas numbers can be generated [3, 5].

Variations of results for the Golden Section include [6, 9]:

$$\lim_{n\to\infty}\frac{F_{n+6}}{F_n}=8\varphi+5, \tag{1.5}$$

and for $\{\varphi\}$, the decimal part of φ:

$$\begin{aligned}\{\phi\} &=\left(1-\{\phi\}\right)^{\frac{1}{2}}\\ &=1-\sum_{r=1}^{\infty}C_r\left(F_r\phi-F_{r+1}\right)\\ &=\phi-1,\end{aligned} \tag{1.6}$$

in which

$$C_r=\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\ldots\left(\frac{3}{2}-r\right)}{r!},$$

so that

$$\varphi=\frac{\left(2+\sum_{r=1}^{\infty}C_rF_{r+1}\right)}{\left(1+\sum_{r=1}^{\infty}C_rF_r\right)}. \tag{1.7}$$

which may be compared with the known [6, 8]:

$$\varphi=\lim_{n\to\infty}\frac{F_{n+1}}{F_n}. \tag{1.8}$$

We note that (1.7) includes an irrational fraction on the right hand side, whereas (1.8) contains a rational fraction. φ may also be expressed as an expansion of the numerical value [6]:

$$\begin{aligned}\varphi&=\tfrac{1}{2}\left(1+\sqrt{5}\right)\\ &=\tfrac{1}{2}+\sqrt{\tfrac{5}{4}}\\ &=\tfrac{1}{2}+(1+0.25)^{\frac{1}{2}}\\ &=1.5+\sum_{r=1}^{\infty}C_r\left(\tfrac{1}{4}\right)^r.\end{aligned} \tag{1.9}$$

Here we consider how these functions contribute to $\{\varphi\}$, the decimal string of φ.

2 Development of $\{\varphi\}$

The progressive sums for Equations (1.6) and (1.9) are displayed in Table 1. For Equation (1.6) the first four decimal places are quickly achieved, but after the changes in the decimal places are relatively slow. As can be expected from such a direct calculation in Equation (1.9) which is not directly related to the Fibonacci numbers, the decimals are obtained almost sequentially with each r. The first seven decimal places are reached by $r = 9$.

The elements of the Lucas sequence also have:

$$\varphi = \lim_{n\to\infty} \frac{L_{n+1}}{L_n} \tag{2.1}$$

The first six decimal places for φ can be obtained from both F_{18}/F_{17} (Equation (1.8)) and L_{17}/L_{16}, and this also applies for $\left(\left(F_{23}/F_{17}\right)-5\right)/8$ from Equation (1.5)

r	C_r	$\varphi = 1.5 + \sum_{r=1}^{\infty} C_r\left(\frac{1}{4}\right)^r$	$\varphi = 2 - \sum_{r=1}^{\infty} C_r\left(F_r\varphi - F_{r+1}\right)$
1	0.5000000	1.6250000	1.6909830
2	–0.1250000	1.6171875	1.6432385
3	0.0625000	1.6181640	1.6284842
4	–0.0390625	1.6180115	1.6227851
5	0.0273437	1.6180382	1.6205979
6	–0.0205078	1.6180332	1.6192004
7	0.0161132	1.6180341	1.6186448
8	–0.0130920	1.6180340	1.6183661
9	0.0109100	1.6180339	1.6182224
10	–0.0092735	1.6180339	1.6181232
11	0.0080089	1.6180339	1.6180838
…	…	…	…
15	0.0049815		1.61803999

Table 1. Comparisons of decimal expansions of φ

If F_{n+1} is the length of a line subdivided into segments F_{n-1} and F_n, then, when

$$F_{n+1}/F_n \cong F_n/F_{n-1} \tag{2.2}$$

defines φ, that is

$$F_{n+1}F_{n-1} \cong F_n^2 \tag{2.3}$$

an approximation to Simson's identity [4], which even when $n = 9$ yields

$$F_{n+1}F_{n-1} = 1155 \cong 1156 = F_n^2$$

and so the equality is approached more closely as n increases. However, as noted above in Equation (1.7), a ratio of irrationals gives a stronger result than a ratio of rationals. One can also infer this in another format by generalising the results in Table 6.5 of Havil [4], namely,

$$\left|\varphi - \frac{F_{n+1}}{F_n}\right| < \frac{1}{F_n^2}.$$

For the ratios F_{n+1}/F_n and L_{n+1}/L_n, the values of n which correspond to the initial appearance of a decimal in the string for φ satisfy an inhomogeneous Pellian-like recurrence relation

$$n_{i+1} = 2n_i - n_{i-1} \pm 1 \tag{2.4}$$

as in Table 2.

Decimals	6	1	8	0	3	3	9	8	8
n_F	4	7	10	12	14	17	19	21	23
n_L	5	6	9	11	14	16	18	21	24

Table 2. Values for (2.4) – *F*: Fibonacci; *L*: Lucas

3 Concluding comments

Bisection of sequences divides sequence in two. This can be further generalized with multisection [2] of series which uses the primitive roots of unity to divide a given series into a number of sections. It is also related to lacunary recurrence relations [8] where there are gaps in the actual recurrence relation, such as.

$$F_n = 2F_{n-1} - F_{n-3}, \ n \geq 4, \tag{3.1}$$

with initial conditions $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, which generates the Fibonacci sequence. It is a third order equation but there are gaps between the two terms on the right hand side, but not within the sequence it generates.

We also note that the structure of the Fibonacci numbers, as analysed within the modular ring Z_5 for instance, prevents the approximation (2.3) from becoming an equality. The exact form is Simson's identity, namely

$$F_{n+1}F_{n-1} = F_n^2 + (-1)^n \tag{3.2}$$

For example,

$$F_{40}F_{38} = 4000054745112195$$

and

$$F_{39} = 4000054745112196,$$

which is also illustrated in Table 3 with the right-end-digits.

n	9	11	17	18	22	29	39	46
$\left(F_n^2\right)^*$	6	1	9	6	1	1	6	9
$\left(F_{n+1}F_{n-1}\right)^*$	5	0	8	7	2	0	5	0

Table 3. Right-end-digits ()* for Simson's identity

References

[1] Atanassov, K., V. Atanassova, A. Shannon, J. Turner. *New Visual Perspectives on the Fibonacci Numbers*. New York: World Scientific, 2002.

[2] Carlitz, L. A binomial Identity Arising from a Sorting Problem, *SIAM Review*. Vol. 6, 1964, 20–30.

[3] Gougenbaum, A. About the Linear Sequence of Integers and that Each Term is the Sum of the Two Preceding. *The Fibonacci Quarterly*. Vol. 9, 1971, 277–295, 298.

[4] Havil, J. *The Irrationals: A Story of the Numbers You Can't Count on*. Princeton & Oxford: Princeton University Press, 2012.

[5] Hoggatt, V. E., Jr. *Fibonacci and Lucas Numbers*. Boston: Houghton-Mifflin, 1969.

[6] Leyendekkers, J. V., A. G. Shannon. 2012. *On the Golden Ratio* (submitted).

[7] Livio, M. *The Golden Ratio*. New York: Golden Books, 2002.

[8] Shannon, A.G. Some Lacunary Recurrence Relations. *The Fibonacci Quarterly*. Vol. 18, 1980, 73–79.

[9] Shannon, A.G., J.V. Leyendekkers. Pythagorean Fibonacci Patterns. *International Journal of Mathematical Education in Science and Technology*. Vol. 43, 2012, 554–559.