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On two new means of two variables

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Abstract: Let A, G and L denote the arithmetic, geometric resp. logarithmic means of two positive number, and let P denote the Seiffert mean. We study the properties of two new means X resp. Y, defined by

$$X = A \cdot e^{G/P-1} \quad \text{and} \quad Y = G \cdot e^{L/A-1}.$$

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1 Introduction

Let a, b be two positive numbers. The logarithmic and identric means of a and b b are defined by

$$\begin{aligned} L &= L(a,b) = \frac{a-b}{\ln a - \ln b}\ (a \neq b), \quad L(a,a) = a; \\ I &= I(a,b) = \frac{1}{e}(b^b/a^a)^{1/(b-a)}\ (a \neq b), \quad I(a,a) = a. \end{aligned} \tag{1.1}$$

The Seiffert mean P is defined by

$$P = P(a,b) = \frac{b-a}{2\arcsin\dfrac{b-a}{a+b}}\ (a \neq b), \quad P(a,a) = a. \tag{1.2}$$

Let

$$A = A(a,b) = \frac{a+b}{2}, \quad G = G(a,b) = \sqrt{ab} \quad \text{and} \quad H = H(a,b) = \frac{2ab}{a+b}$$

denote the classical arithmetic, geometric, resp. harmonic means of a and b. There exist many papers which study properties of these means. We quote e.g. [1], [2] for the identric and logarithmic means, and [3] for the mean P.

The means L, I and P are particular cases of the "Schwab-Borchardt mean", see [4], [5] for details. The means of two arguments have important applications also in number- theoretical problems. For example, the solution of certain conjectures on prime numbers in paper [11] is based on the logarithmic mean L.

The aim of this paper is the study of two new means, which we shall denote by $X = X(a, b)$ and $Y = Y(a, b)$, defined as follows:

$$X = A \cdot e^{\frac{G}{P}-1}, \tag{1.3}$$

resp.

$$Y = G \cdot e^{\frac{L}{A}-1}. \tag{1.4}$$

Clearly $X(a, a) = Y(a, a) = a$, but we will be mainly interested for properties of these means for $a \neq b$.

2 Main results

Lemma 2.1. *The function $f(t) = te^{\frac{1}{t}-1}$, $t > 1$ is strictly increasing. For all $t > 0$, $t \neq 0$ one has $f(t) > 1$. For $0 < t < 1$, f is strictly decreasing. As a corollary, for all $t > 0$, $t \neq 1$ one has*

$$1 - \frac{1}{t} < \ln t < t - 1. \tag{2.1}$$

Proof. As $\ln f(t) = \ln t + \dfrac{1}{t} - 1$, we get

$$\frac{f'(t)}{f(t)} = \frac{t-1}{t^2},$$

so $t_0 = 1$ will be a minimum point of $f(t)$, implying $f(t) \geq f(1) = 1$, for any $t > 0$, with equality only for $t = 1$. By taking logarithm, the left side of (2.1) follows. Putting $1/t$ in place of t, the left side of (2.1) implies the right side inequality. □

Theorem 2.1. *For $a \neq b$ one has*

$$G < \frac{A \cdot G}{P} < X < \frac{A \cdot P}{2P - G} < P. \tag{2.2}$$

Proof. Applying (2.1) for $t = \dfrac{X}{A}$ ($\neq 1$, as $G \neq P$ for $a \neq b$), and by taking into account of (1.3) we get the middle inequalities of (2.2). As it is well known that (see [3])

$$\frac{A+G}{2} < P < A, \tag{2.3}$$

the first inequality of (2.3) implies the last one of (2.2), while the second inequality of (2.3) implies the first one of (2.2).

In a similar manner, the following is true:

Theorem 2.2. *For $a \neq b$ one has*

$$H < \frac{L \cdot G}{A} < Y < \frac{G \cdot A}{2A - L} < G. \tag{2.4}$$

Proof. Applying (2.1) for $t = \frac{Y}{G}$ by (1.4) we can deduce the second and third inequalities of (2.4). Since $H = \frac{G^2}{A}$, the first and last inequality of (2.4) follows by the known inequalities (see e.g. [1] for references)

$$G < L < A. \tag{2.5}$$

□

The second inequality of (2.2) can be strongly improved, as follows:

Theorem 2.3. *For $a \neq b$ one has*

$$1 < \frac{L^2}{G \cdot I} < \frac{L}{G} \cdot e^{\frac{G}{L} - 1} < \frac{X \cdot P}{A \cdot G}. \tag{2.6}$$

Proof. As $L < P$ (due to Seiffert; see [3] for references) and

$$f\left(\frac{P}{G}\right) = X \cdot \frac{P}{A \cdot G}, \tag{2.7}$$

where f is defined in Lemma 2.1, and by taking into account of the inequality (see [1])

$$\frac{L}{I} < e^{\frac{G}{L} - 1}, \tag{2.8}$$

by the monotonicity of f one has

$$f\left(\frac{P}{G}\right) = \frac{L}{G} \cdot e^{\frac{G}{L} - 1} > \frac{L}{G} \cdot \frac{L}{I} = \frac{L^2}{G \cdot I}. \tag{2.9}$$

By an inequality of Alzer (see [1] for references) one has

$$L^2 > G \cdot I, \tag{2.10}$$

thus all inequalities of (2.6) are established.

The following estimates improve the left side of (2.4):

Theorem 2.4. *For $a \neq b$,*

$$H < \frac{G^2}{I} < \frac{L \cdot G}{A} < \frac{G \cdot (A+L)}{3A - L} < Y. \tag{2.11}$$

Proof. Since $H = \dfrac{G^2}{A}$, the first inequality of (2.11) follows by the known inequality $I < A$ (see [1] for references). The second inequality of (2.11) follows by another known result of Alzer (see [1] for references, and [3] for improvements)

$$A \cdot G < L \cdot I. \tag{2.12}$$

Finally, to prove the last inequality of Y, remark that the logarithmic mean of Y and G is

$$L(Y, G) = \frac{Y - G}{\ln Y/G} = \frac{(G - Y)A}{A - L}. \tag{2.13}$$

Now, by the right side of (2.5) applied to $a = Y$, $b = G$ we have

$$L(Y, G) < (Y + G)/2,$$

so

$$2A(G - Y) < (A - L)(Y + G),$$

which after some transformations gives the desired inequality. □

Similarly to (2.11) we can state:

Theorem 2.5.

$$\frac{A \cdot G}{P} < \frac{A(P + G)}{3P - G} < X. \tag{2.14}$$

Proof. $L(X, A) = (X - A) \log X/A = \dfrac{(A - X)P}{P - G} < \dfrac{X + A}{2}$, so after simple computations we get the second inequality of (2.14). The first inequality becomes

$$(P - G)^2 > 0.$$ □

A connection between the two means X and Y is provided by:

Theorem 2.6. *For $a \neq b$,*

$$A^2 \cdot Y < P \cdot L \cdot X. \tag{2.15}$$

Proof. By using the inequality (see [3])

$$\frac{A}{L} < \frac{P}{G}, \tag{2.16}$$

and remarking that

$$f\left(\frac{A}{L}\right) = \frac{A}{L \cdot G} \cdot Y, \tag{2.17}$$

by the monotonicity of f one has $f\left(\frac{A}{L}\right) < f\left(\frac{P}{G}\right)$, so by (2.7) and (2.17) we can deduce inequality (2.15). □

Remark 2.1. By the known identity (see [1], [2])

$$\frac{I}{G} = e^{\frac{A}{L}-1} \tag{2.18}$$

and the above methods one can deduce the following inequalities (for $a \neq b$):

$$1 < \frac{L \cdot I}{A \cdot G} < \frac{G}{P} \cdot e^{\frac{P}{G}-1}. \tag{2.19}$$

Indeed, as $f\left(\frac{L}{A}\right) = \frac{L}{A} \cdot e^{\frac{A}{L}-1} = \frac{L \cdot I}{A \cdot G} > 1$ we reobtain inequality (2.12). On the other hand, by (2.16) we can write, as $1 > \frac{L}{A} > \frac{G}{P}$ that $f\left(\frac{L}{A}\right) < f\left(\frac{G}{P}\right)$ i.e. the complete inequality (2.19) is established.

Theorem 2.7. *For $a \neq b$*

$$X < A\left[\frac{1}{e} + \left(1 - \frac{1}{e}\right)\frac{G}{P}\right] \tag{2.20}$$

and

$$Y < G\left[\frac{1}{e} + \left(1 - \frac{1}{e}\right)\frac{L}{A}\right]. \tag{2.21}$$

Proof. The following auxiliary result will be used:

Lemma 2.2. *For the function f of Lemma 2.1, for any $t > 1$ one has*

$$f(t) < \frac{1}{e}(t + e - 1) \tag{2.22}$$

and

$$f(t) < \frac{1}{e}\left(t + \frac{1}{2t} + e - \frac{3}{2}\right) < \frac{1}{e}(t + e - 1). \tag{2.23}$$

Proof. By the series expansion of e^x and by $t > 1$, we have

$$f(t) = \frac{1}{e}\left(t + 1 + \frac{1}{2t} + \frac{1}{3!t^2} + \frac{1}{4!t^3} + \dots\right)$$

$$= \frac{1}{e}\left(t + \frac{1}{1!} + \frac{1}{2!} + \dots\right) = \frac{1}{e}(t + e - 1),$$

so (2.22) follows.

Similarly,

$$f(t) = \frac{1}{e}\left(t + 1 + \frac{1}{2t} + \frac{1}{3!t^2} + \frac{1}{4!t^3} + \dots\right)$$
$$< \frac{1}{e}\left[t + 1 + \frac{1}{2t} + e - \left(1 + \frac{1}{1!} + \frac{1}{2!}\right)\right] = \frac{1}{e}\left(t + \frac{1}{2t} + e - \frac{3}{2}\right),$$

so (2.23) follows as well.

Now, (2.20) follows by (2.22) and (2.7), while (2.21) by (2.23) and (2.17). □

Theorem 2.8. *For $a \neq b$ one has*

$$P^2 > A \cdot X \tag{2.24}$$

Proof. Let $x \in \left(0, \frac{\pi}{2}\right)$. In the recent paper [7] we have proved the following trigonometric inequality:

$$\ln \frac{x}{\sin x} < \frac{\sin x - \cos x}{2 \sin x}. \tag{2.25}$$

Remark that by (1.2) one has

$$P(1 + \sin x, 1 - \sin x) = \frac{\sin x}{x},$$

$$A(1 + \sin x, 1 - \sin x) = 1, \quad G(1 + \sin x, 1 - \sin x) = \cos x,$$

so (2.24) may be rewritten also as

$$P^2(1 + \sin x, 1 - \sin x) > A(1 + \sin x, 1 - \sin x) \cdot X(1 + \sin x, 1 - \sin x). \tag{2.26}$$

For any $a, b > 0$ one can find $x \in \left(0, \frac{\pi}{2}\right)$ and $k > 0$ such that

$$a = (1 + \sin x)k, \quad b = (1 - \sin x)k.$$

Indeed, let $k = \frac{a+b}{2}$ and $x = \arcsin \frac{a-b}{a+b}$.

Since the means P, A and X are homogeneous of order one, by multiplying (2.26) by k, we get the general inequality (2.24). □

Corollary 2.1.

$$P^3 > \frac{A^2 L^2}{I} > A^2 G. \tag{2.27}$$

Proof. By (2.6) of Theorem 2.3 and (2.24) one has

$$\frac{L^2 A}{I \cdot P} < X < \frac{P^2}{A}, \tag{2.28}$$

so we get $P^3 > \frac{A^2 L^2}{I} > A^2 G$ by inequality (2.10). □

Remark 2.2. Inequality (2.27) offers an improvement of

$$P^3 > A^2 G \tag{2.29}$$

from paper [3]. We note that further improvements, in terms of A and G can be deduced by the "sequential method" of [3]. For any application of (2.29), put $a = 1 + \sin x$, $b = 1 - \sin x$ in (2.29) to deduce

$$\frac{\sin x}{x} > \sqrt[3]{\cos x}, \quad x \in \left(0, \frac{\pi}{2}\right), \tag{2.30}$$

which is called also the Mitrinović–Adamović inequality (see [6]).

Since (see [3])

$$P < \frac{2A + G}{3}, \tag{2.31}$$

by the above method we can deduce

$$\frac{\sin x}{x} < \frac{\cos x + 2}{3}, \tag{2.32}$$

called also as the Cusa–Huygens inequality. For details on such trigonometric or related hyperbolic inequalities, see [6].

Theorem 2.9. *One has the inequalities*

$$P^2 > \sqrt[3]{A^2 \left(\frac{A+G}{2}\right)^4} > A \cdot X, \tag{2.33}$$

Proof. The first inequality of (2.33) appeared in our paper [3]. For the second inequality, consider the application

$$f(x) = \ln \frac{\cos x + 1}{2} - \frac{3}{4}\left(x \cot x - 1\right).$$

An easy computation gives $4(\sin x)^2(\cos x + 1)f'(x) = 3x + 3x\cos x - (\sin x)^3 - 3\sin x - 3\sin x \cos x = g(x)$. Here $g'(x) = 3\sin x \cdot h(x)$, where $h(x) = 2\sin x - x - \sin x \cos x$. One has $h'(x) = 2(\cos x)(1 - \cos x) > 0$, so $h(x) > h(0) = 0$. This in turn gives $g(x) > g(0)$ implying $f'(x) > 0$. Therefore, $f(x) > f(0) = 0$ for $x \in \left(0, \frac{\pi}{2}\right)$. This proves the second inequality of (2.33). □

Remark 2.3. From the second inequalit of (2.14) it is immediate that the following weaker inequality holds:

$$X > \frac{2G + A}{3}, \tag{2.34}$$

This follows by $P < (2A + G)/3$ (See [3]). Since $L < (2G + A)/2$, we get

$$X > \frac{2G + A}{3} > L, \tag{2.35}$$

so we can deduce by (2.33) a chain of inequalities for P, which improves a result from our paper [8].

Theorem 2.10. *One has*

$$X > A + G - P, \tag{2.36}$$

Proof. By using the notations from the proof of Theorem 2.8, and by taking logarithms, the inequality becomes

$$f(x) = x \cot x - 1 - \ln(1 + \cos x - \sin x/x)) > 0.$$

After elelemtary computations one finds that the sign of derivative of f depends on the sign of the function $g(x) = \dfrac{x}{\sin x} + (\sin x)^2 - (x^2)\cos x - x^2$. To prove that $g(x) > 0$, we have to show that

$$F(x) = x \sin x + (\sin x)^2 - (x^2)(\cos x) - x^2 > 0.$$

We get $F'(x) = \sin x - x\cos x + 2\sin x \cos x + x^2 \cos x - 2x$ and $F''(x) = 3x \sin x + x^2 \cos x - 4(\sin x)^2$. We will prove that $F''(x) > 0$, or equivalently $4t^2 - t - \cos x < 0$, where $t = \sin x/x$. By solving the above quadratic inequality, and by taking into account of the Cusa-Huygens inequality $t < (2+u)/3$, where $u = \cos x$, we have to prove the following relation: $(2+u)/3 < [3 + \sqrt{9 + 16u}]/8$; or after some computations, with $(2u+1)(u-1) < 0$, which is true. Since $F''(x) > 0$, we get $F'(x) > F'(0) = 0$, so $F(x) > F(0) = 0$. The function f being strictly increasing, the result follows, as $f(0+) = 0$. □

Remark 2.4. Inequality (2.36) is slightly stronger than the right side of (2.14). Indeed, after some transformations, this becomes

$$3P^2 - 2P \cdot (A + 2G) + 2AG + G^2 < 0.$$

Resolving this quadratic inequality, this becomes $[3P - (2A + G)].(P - G) < 0$, which is true by $G < P < (2A + G)/3$.

Theorem 2.11.

$$P \cdot X < \left(\frac{A+G}{2}\right)^2. \tag{2.37}$$

Proof. It is immediate that we have to prove the following inequality:

$$f(x) = \ln(x/\sin x) - 2\ln[2/(1 + \cos x)] - (x \cot x - 1) > 0.$$

After computations we get that the sign of $f'(x)$ depends on the sign of $g(x) = (\sin x)(1 + \cos x)(\sin x - x\cos x) - (x \sin x \cos x)(1 + \cos x) = (x^2)(1 + \cos x) + 2x(\sin x)^3$. By $1 = \cos x = 2[\cos(x/2)]^2$ and $\sin x = 2\sin(x/2)\cos(x/2)$ we can write $g(x) = 2[\cos(x/2)]^2 h(x)$, where $h(x) = (\sin x)^2 - 2x \sin x \cos x + x^2 + 8x\cos(x/2)[\sin(x/2)]^3$. We can deduce $h'(x)/4[\sin(x/2)]^2 = 7x[\cos(x/2)]^2 + 2\sin(x/2)\cos(x/2) - x[\sin(x/2)]^2$. As $\cos(x/2) > \sin(x/2)$, we get $h'(x) > 0$. Thus we have $h(x) > h(0) = 0$, so $g(x) > 0$ and finally, $f'(x) > 0$. The result follows by the remark that $f(x) > f(0+) = 0$. □

Remark 2.5. Relation (2.37) combine with the weaker inequality of (2.14) shows that, $\sqrt{(P \cdot X)}$ lies between the geometric and arithmetic means of A and G.

Remark 2.6. In a recent paper, B. A. Bhayo and the author [9] have proved the following counterpart of relation (2.24):

Theorem 2.12. *The following inequality holds true:*

$$P < (X^c) \cdot (A^{1-c}),$$

where $c = \ln(\pi/2)$.

Remark 2.7. An earlier version of this work appeared in the last paragraph of the arXiv paper [10].

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