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On generalized multiplicative perfect numbers

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Abstract: In this paper we define T^*T multiplicative divisors function. This notion leads us to generalized multiplicative perfect numbers like T^*T perfect numbers, $k - T^*T$ perfect numbers and T^*_0T–super-perfect numbers. We attempt to characterize these numbers.
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1 Introduction

A natural number n is said to be perfect number if it is equal to the sum of its proper divisors. If σ denotes the sum of divisors, for any perfect number n, $\sigma(n) = 2n$. The Euclid–Euler theorem gives the form of even perfect numbers in the form $n = 2^{p-1}(2^p - 1)$, where $2^p - 1$ is a Mersenne prime. Moreover n is said to be super-perfect if $\sigma(\sigma(n)) = 2n$. The Suryanarayana–Kanold theorem gives the general form of even super-perfect numbers $-n = 2^k$, where $2^{k+1} - 1$ is prime. No odd super-perfect numbers are known. For new proofs of these results, see [5, 9]. A divisor d of a natural number n is said to be unitary divisor if $\left(d, \frac{n}{d}\right) = 1$ and n is unitary perfect if $\sigma^*(n) = 2n$, where σ^* denotes the sum of unitary divisors of n. The notion of unitary perfect numbers was introduced M. V. Subbarao and L. J. Waren in 1966, [8]. Five unitary even perfect numbers are known and it is true that no unitary perfect numbers of the form $2^m s$ where s is a square free odd integer [3]. Sándor in [6] introduced the concept of multiplicatively divisor function $T(n)$ and multiplicatively perfect and super-perfect numbers and characterized them. If $T(n)$ denote the product of all divisors of n, then

$$T(n) = \prod_{d|n} d = n^{\frac{\tau(n)}{2}},$$

where $\tau(n)$ is the number of divisors of n. The number $n > 1$ is multiplicatively perfect (or shortly m-perfect) if $T(n) = n^2$, and multiplicatively super-perfect (m-super-perfect), if $T(T(\text{n})) = n^2$. In [1], Antal Bege introduced the concept of unitary divisor function $T^*(n)$ and unitary perfect and super-perfect numbers and characterized them multiplicatively. Let $T^*(n)$ denote the product of all unitary divisors of n:

$$T^*(n) = \prod_{d|n} d = n^{\frac{\tau^*(n)}{2}},$$

where $\left(d, \frac{n}{d}\right) = 1$ and $\tau^*(n)$ is the number of unitary divisors of n. The number $n > 1$ is multiplicatively unitary perfect (or shortly m-unitary-perfect) if $T^*(n) = n^2$, and multiplicatively unitary super-perfect (m-unitary-super-perfect), if $T^*(T^*(n)) = n^2$. It is to be noted that there are no m-super-perfect and m-unitary-super perfect numbers.

2 *T*T*-perfect numbers

Definition 2.1. Let $[T^*T](n)$ or $[TT^*](n)$ denote the product of $T(n)$ and $T^*(n)$, i.e. $[T^*T](n) = T^*(n)T(n)$. Let us call the number $n > 1$ as T^*T–perfect number if $[T^*T](n) = n^2$.

Theorem 2.1. For $n > 1$ there are no T^*T–perfect numbers for non-prime n.

Proof: Let $n = p_1^{\alpha_1} p_2^{\alpha_2} ......p_r^{\alpha_r}$ be the prime factorisation of $n > 1$. It is well-known that

$$\tau(n) = (\alpha_1 + 1)(\alpha_2 + 1)\ldots(\alpha_r + 1) \quad (2.1)$$

and

$$\tau^*(n) = 2^{\omega(n)} = 2^r, \quad (2.2),$$

where $\omega(n)$ is the number of distinct prime divisors of n.

$$[T^*T](n) = T^*(n)T(n) = n^{\frac{\tau(n)}{2}}.n^{\frac{\tau^*(n)}{2}} = n^{\frac{\tau(n)+\tau^*(n)}{2}}$$

For T^*T–perfect numbers

$$2^r + (\alpha_1 + 1)(\alpha_2 + 1)\ldots(\alpha_r + 1) = 4.$$

Since $r \geq 1$, we can have only $(\alpha_1 + 1) = 2$ and $r = 1$, giving $n = p_1$. There are no other solutions $n > 1$ ($n = 1$ is a trivial solution) of the equation.

Thus primes are T^*T–perfect numbers.

For any $n \geq 2$ we have $\tau(n) \geq 2$, so $T(n) \geq 2$.

If n is not a prime, then it is immediate that $\tau(n) \geq 3$, giving

$$T(n) \geq n^{\frac{3}{2}} \quad (2.3)$$

If n is not a prime, then

$$T^*(n) \geq n \quad (2.4)$$

Now relations (2.3) and (2.4) together give $[T^*T](n) \geq n^{\frac{5}{2}}$, where n is not a prime.

Thus, by $\frac{5}{2} > 2$, there are no T^*T–perfect number for non prime n. □

Corollary 2.2. Perfect numbers are not T^*T–perfect numbers.

3 *k–T*T*–perfect numbers

In a similar manner, one can define k–T^*T–perfect numbers by $[T^*T](n) = n^k$, where $k \geq 2$ is given. Since the equation $2^r + (\alpha_1 + 1)(\alpha_2 + 1)\ldots(\alpha_r + 1) = 2k$ has a finite number of solutions, the general form of k–T^*T–perfect numbers can be determined. We present certain particular cases in the following result.

Theorem 3.1.

(i) All tri–T^*T–perfect numbers have the form $n = p_1^3$;

(ii) All 4–T^*T–perfect numbers have the form $n = p_1 p_2$ or $n = p_1^5$;

(ii) All 5–T^*T–perfect numbers have the form $n = p_1^2 p_2$ or $n = p_1^7$;

(iv) All 6–T^*T–perfect numbers have the form $n = p_1^3 p_2$ or $n = p_1^9$;

(v) All 7–T^*T–perfect numbers have the form $n = p_1^4 p_2$ or $n = p_1^{11}$;

(vi) All 8–T^*T–perfect numbers have the form $n = p_1 p_2 p_3$ or $n = p_1^5 p_2$ or $n = p_1^3 p_2^2$ or $n = p_1^{13}$;

(vii) All 9–T^*T–perfect numbers have the form $n = p_1^6 p_2$ or $n = p_1^{15}$;

(viii) All 10–T^*T–perfect numbers have the form $n = p_1^2 p_2 p_3$ or $n = p_1^7 p_2$ or $n = p_1^3 p_2^3$ or $n = p_1^{17}$;

(ix) All 11–T^*T–perfect numbers have the form $n = p_1^5 p_2^2$ or $n = p_1^8 p_2$ or $n = p_1^{19}$;

(x) All 12–T^*T–perfect numbers have the form $n = p_1^3 p_2 p_3$ or $n = p_1^9 p_2$ or $n = p_1^4 p_2^3$ or $n = p_1^{21}$, etc.

Here p_i denote certain distinct primes. We prove only the cases (vi) and (x).

Proof: (vi) For the 8–T^*T–perfect number n, $[T^*T](n) = n^8$, so we must solve the equation

$$2^r + (\alpha_1 + 1)(\alpha_2 + 1)\ldots(\alpha_r + 1) = 16$$

in α_r and r. It is easy to see that the following four cases are possible:

(I) $r = 1,\ \alpha_1 + 1 = 14$;

(II) $r = 2,\ \alpha_1 + 1 = 4, \alpha_2 + 1 = 3$;

(III) $r = 2,\ \alpha_1 + 1 = 6, \alpha_2 + 1 = 2$;

(IV) $r = 3,\ \alpha_1 + 1 = 2,\ \alpha_2 + 1 = 2,\ \alpha_3 + 1 = 2$.

This gives the general forms of all 8–T^*T–perfect numbers, namely:

$(r = 1, \alpha_1 = 13)$ $n = p_1^{13}$;

$(r = 2, \alpha_1 = 3, \alpha_2 = 2)$ $n = p_1^3 p_2^2$;

$(r = 2, \alpha_1 = 5, \alpha_2 = 1)$ $n = p_1^5 p_2$;

$(r = 3, \alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 1)$ $n = p_1 p_2 p_3$.

(x) To find the general form of 12–$T*T$–perfect numbers, we must solve the equation

$$2^r + (\alpha_1 + 1)(\alpha_2 + 1)\ldots(\alpha_r + 1) = 24$$

in α_r and r. It is easy to see that the following four cases are possible:

(I) $r = 1, \alpha_1 + 1 = 21$;

(II) $r = 2, \alpha_1 + 1 = 4, \alpha_2 + 1 = 5$;

(III) $r = 2, \alpha_1 + 1 = 10, \alpha_2 + 1 = 2$;

(IV) $r = 3, \alpha_1 + 1 = 3, \alpha_2 + 1 = 2, \alpha_3 + 1 = 2$.

Thus the general forms of all 12–$T*T$–perfect numbers are namely:

$(r = 1, \alpha_1 = 21)$ $n = p_1^{21}$;

$(r = 2, \alpha_1 = 3, \alpha_2 = 4)$ $n = p_1^3 p_2^4$;

$(r = 2, \alpha_1 = 9, \alpha_2 = 1)$ $n = p_1^9 p_2$;

$(r = 3, \alpha_1 = 2, \alpha_2 = 1, \alpha_3 = 1)$ $n = p_1^2 p_2 p_3$. □

Corollary 3.2. (i) There are no perfect numbers which are tri–$T*T$–perfect number.

(ii) $n = 6$ is the only perfect number which is 4–$T*T$–perfect number.

(iii) $n = 28$ is the only perfect number which is 5–$T*T$–perfect number.

(iv) $n = 496$ is the only perfect number which is 7–$T*T$–perfect number.

(v) $n = 8128$ is the only perfect number which is 9–$T*T$–perfect number.

Theorem 3.3. Let p be a prime, with $2^p - 1$ being a Mersenne prime. Then $n = 2^{p-1}(2^p - 1)$ is the only perfect number which is a $(p + 2)$–$T*T$–perfect number.

Proof: If $n = 2^{p-1}(2^p - 1)$ is an even perfect number, then $\tau(n) = 2p$, $\omega(n) = 2$, $\tau^*(n) = 4$, and so

$$[T*T](n) = n^{\frac{\tau(n)+\tau^*(n)}{2}} = n^{\frac{2p+4}{2}} = n^{p+2}.$$ □

4 $T*_0T$–super-perfect and k–$T*_0T$–perfect numbers

Definition 4.1: The number $n > 1$ is a $T*_0T$–super-perfect number if $T*(T(n)) = n^2$, and k–$T*_0T$–perfect number if $T*(T(n)) = n^k$, where $k \geq 3$.

Theorem 4.2. All $T*_0T$–super-perfect numbers have the form $n = p_1^3$, where p_1 is an arbitrary prime.

Proof: First, we determine $T*(T(n))$:

$$T^*(T(n)) = (T(n))^{\frac{\tau^*(T(n))}{2}} = (n^{\frac{\tau(n)}{2}})^{\frac{\tau^*(T(n))}{2}} \tag{4.1}$$

$$\tau^*(T(n)) = \tau^*(n^{\frac{\tau(n)}{2}}) = \tau^*(n) \tag{4.2}$$

From (4.1) and (4.2), $T^*(T(n)) = n^{\frac{\tau(n).\tau^*(n)}{4}}$. By using the relations (2.1) and (2.2), for $T^*{}_0T$–super-perfect numbers

$$2^r (\alpha_1 + 1) (\alpha_2 + 1)\dots(\alpha_r + 1) = 8.$$

Since $r \geq 1$, we can have only $\alpha_1 + 1 = 4$ and $r = 1$, implying $r = 1, \alpha_1 = 3$, i.e. $n = p_1^3$. In a similar manner k–$T^*{}_0T$–perfect numbers can be defined. Since the equation

$$2^r (\alpha_1 + 1) (\alpha_2 + 1)\dots(\alpha_r + 1) = 4k$$

has a finite number of solutions, the general form of k–$T^*{}_0T$–perfect numbers can be determined. □

Theorem 4.3.

(i) All tri–$T^*{}_0T$–perfect numbers have the form $n = p_1^5$;

(ii) All 4–$T^*{}_0T$–perfect numbers have the form $n = p_1 p_2$ or $n = p_1^7$;

(iii) All 5–$T^*{}_0T$–perfect numbers have the form $n = p_1^9$;

(iv) All 6–$T^*{}_0T$–perfect numbers have the form $n = p_1^2 p_2$ or $n = p_1^{11}$;

(v) All 7–$T^*{}_0T$–perfect numbers have the form $n = p_1^{13}$;

(vi) All 8–$T^*{}_0T$–perfect numbers have the form $n = p_1^3 p_2$ or $n = p_1^{15}$;

(vii) All 9–$T^*{}_0T$–perfect numbers have the form $n = p_1^2 p_2^2$ or $n = p_1^{17}$;

(viii) All 10–$T^*{}_0T$–perfect numbers have the form $n = p_1^4 p_2$ or $n = p_1^{19}$;

Proof: We prove only the case (viii). For 10–$T^*{}_0T$–perfect number $T^*(T(n)) = n^{10}$. We must solve the equation

$$2^r (\alpha_1 + 1) (\alpha_2 + 1)\dots(\alpha_r + 1) = 40$$

in r and α_r. It is easy to see that the following cases are possible:

(I) $r = 1$, $\alpha_1 + 1 = 20$.

(II) $r = 2$, $\alpha_1 + 1 = 5$, $\alpha_2 + 1 = 2$.

This gives the general form of all 10–$T^*{}_0T$–perfect numbers, namely:

$(r = 2, \alpha_1 = 4, \alpha_2 = 1)$ $n = p_1^4 p_2$;

$(r = 1, \alpha_1 = 19)$ $n = p_1^{19}$. □

Theorem 4.4. Let p be a prime, with $2^p - 1$ being a Mersenne prime. Then $2^{p-1}(2^p - 1)$ is the only perfect number, which is $2p$–$T^*{}_0T$–perfect number.

Proof: By writing $2^r (\alpha_1 + 1) (\alpha_2 + 1)\dots(\alpha_r + 1) = 8p$ (where p is prime), the following cases are only possible:

(i) $r = 2, \alpha_1 + 1 = 2$, $\alpha_2 + 1 = p$

(ii) $r = 1$, $\alpha_1 + 1 = 4p$

Then $n = p_1 p_2{}^{p-1}$ or $n = p_1{}^{4p-1}$ are the general form of $2p$–T^*_0T–perfect numbers. By the Euler–Euclid theorem, $p_1 p_2^{p-1} = 2^{p-1}(2^p - 1)$ iff $p_1 = 2^p - 1$ and $p_2 = 2$. □

5 *k*–T_0T^*–perfect numbers

Definition 5.1. The number $n > 1$ is a k–T_0T^*–perfect number (where $k \geq 2$) if $T(T^*(n)) = n^k$.

First, we determine $T(T^*(n))$. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ be the prime factorisation of $n > 1$, then $\tau^*(n) = 2^r$ and $T^*(n) = n^{2^{r-1}}$.

$$T(T^*(n)) = (T^*(n))^{\frac{\tau(T^*(n))}{2}} = \left(n^{2^{r-1}}\right)^{\frac{\tau\left(n^{2^{r-1}}\right)}{2}} = n^{\frac{2^r \tau\left(n^{2^{r-1}}\right)}{4}} \tag{5.1}$$

Since $n^{2^{r-1}} = p_1^{\alpha_1 2^{r-1}} p_2^{\alpha_2 2^{r-1}} \dots p_r^{\alpha_r 2^{r-1}}$ and $\tau(n)$ is a multiplicative function, so

$$\tau(p_i^{\alpha_i 2^{r-1}}) = \alpha_i 2^{r-1} + 1; \; i = 1, 2, \dots, r \tag{5.2}$$

From the relations (5.1) and (5.2) for k–T_0T^*–perfect number

$$2^r (\alpha_1 2^{r-1} + 1)(\alpha_2 2^{r-1} + 1) \dots (\alpha_r 2^{r-1} + 1) = 4k \tag{5.3}$$

Solving the equation (5.3) in r and α_r, we can determine forms of the k–T_0T^*–perfect numbers.

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