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$(2, d)$-Sigraphs

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Abstract: An edge uv in a graph G is *directionally labeled* by an ordered pair ab if the label $\ell(uv)$ on uv is ab in the direction from u to v, and $\ell(vu) = ba$. A $(2, d)$-*sigraph* $G = (V, E)$ is a graph in which every edge is directionally labeled by an ordered pair $ab \in \{++, --, +-, -+\}$. A $(2, d)$-sigraph G has a *uniform-directional edge labling*(*ude-labeling*) at a vertex u in G, if for each neighbor v of u, either $\ell(uv) \in \{++, +-\}$ or $\ell(uv) \in \{--, -+\}$. Further, G is *ude-balanced* if it has such a labeling at each of its vertex. Two characterizations of ude-balanced $(2, d)$-sigraphs are obtained. Using a notion of 2-splitting of a $(2, d)$-sigraphs, we define a 2-balanced $(2, d)$-sigraph, and obtain a characterization of 2-balanced $(2, d)$-sigraph which is similar to a characterization of balanced sigraphs. Further, the notion of clusterability of signed graphs is extended to $(2, d)$-sigraphs, and a characterization of clusterable $(2, d)$-sigraph is obtained. The notions of ude-balance and clusterability are extended to (n, d)-sigraphs. Some applications of $(2, d)$-sigraphs are also mentioned.
Keywords: Signed graph, Directional adjacency, $(2, d)$-sigraph, Uniform directional labeling, Clusterability, Bidirected graph.
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1 Introduction

For any definition on graphs we refer the book [4].

A *signed graph* (or simply, a *sigraph*) $G = (V, E)$ is a graph in which every edge is signed $+$ or $-$, and the labels on the edges are called *signs* of the edges.
A sigraph G is said to be *balanced* [3] if every cycle in G has an even number of edges signed $-$.

The following results are well known.

Proposition 1. (F. Harary [3]) *A signed graph $G = (V, E)$ is balanced if, and only if, it is possible to divide its vertex set V into two disjoint subsets V_1 and V_2, one of them possibly empty, such that $V = V_1 \cup V_2$, every positive edge joins two vertices in V_1 or in V_2, and every negative edge joins a vertex in V_1 and a vertex in V_2.*

A *marked graph* is a graph in which every vertex is labeled $+$ or $-$ and the labels are called *marks* of the vertices. There is a vast literature on signed and marked graphs (see for example [12]). One of them relates marked graphs with signed graphs as follows.

Proposition 2. (E. Sampathkumar [5]) *A signed graph G is balanced if, and only if, it is possible to mark its vertices with $+$ and $-$ such that the sign of every edge in G is the product of the marks of its ends.*

2 Directional labeling of an edge and Uniform-directional-edge-labeling of a graph

Sometimes it may be necessary to distinguish the adjacency between u and v in the directions from u to v, and from v to u, independently. For example, we can consider the adjacency from u to v as positive, and that from v to u as negative.
Suppose, for example,
• A, B are two persons.
(a) A is talking to B.
(b) A is a boss and B is a subordinate of A.
• A, B are nodes in an electrical network, and current is flowing from A to B.
• A is a transmitter and B is a receiver.
• There is a one way road from a place A to a place B.
• A and B are two persons who are in contact with each other, and A likes this contact, whereas B dislikes this contact.

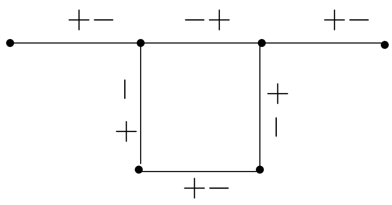
In all the above cases, we can consider the adjacency from A to B as positive, and that from B to A as negative. In general, therefore, an edge uv in a graph G can be *directionally labeled* by an ordered pair ab if the label $\ell(uv)$ on uv is ab in the direction from u to v, and $\ell(vu) = ba$. This motivates the following definition.

Definition 3. A $(2, d)$-*sigraph* (G, ℓ) is a graph $G = (V, E)$ in which every edge is directionally labeled by the function ℓ, called a *directional edge-labeling* of G, assigning an ordered pair $ab \in \{++, --, +-, -+\}$ to each edge of G.

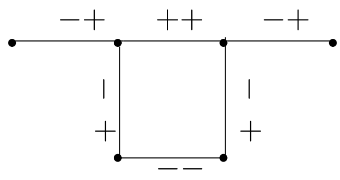
Definition 4. A $(2, d)$-sigraph (G, ℓ) has *uniform-directional edge-labling(ude-labeling)* at a vertex u if for each neighbor v of u, either $\ell(uv) \in \{++, +-\}$ or $\ell(uv) \in \{--, -+\}$. Further, G is *ude-balanced* if it has such a labeling at each of its vertex.

If $\ell(uv) \in \{++, +-\}$ for each neighbor v of u, we indicate this fact by $\ell_1(u) = +$. Similarly, if $\ell(uv) \in \{--, -+\}$, we indicate this fact by $\ell_1(u) = -$. Thus, if a $(2, d)$-sigraph is *ude-balanced*, then for each vertex u in G, either $\ell_1(u) = +$ or $\ell_1(u) = -$.

Hence, in a $(2, d)$-sigraph (G, ℓ) the adjacency between two vertices is regarded as *directional adjacency*, as illustrated in the example above.



G_1: ude-balanced



G_2: Not ude-balanced

Figure 1

Clearly, every graph G possesses a *trivial* uniform-directional edge-labeling ℓ, viz., one in which $\ell(e) = ++ \ \forall\, e \in E(G)$ or $\ell(e) = -- \ \forall\, e \in E(G)$. Further, we have the following fact too.

Proposition 5. *Every graph $G = (V, E)$ has a nontrivial edge-labeling from the set $\{++, --, +-, -+\}$ such that for each vertex u in G, either $\ell_1(u) = +$, or $\ell_1(u) = -$.*

Proof. Let $G = (V, E)$ be any graph having at least one edge. Let $V = V_1 \cup V_2$ be a partition of V. Label all edges in V_1 with label $++$, and label all edges in V_2 with $--$, also label each edge from V_1 to V_2 directionally with label $+-$. Then for each vertex u in the $(2, d)$-sigraph thus obtained, either $\ell_1(u) = +$ or $\ell_1(u) = -$. □

We now obtain a characterization of a ude-balanced $(2, d)$-sigraph (G, ℓ) which is very much similar to Proposition 1.

Proposition 6. *For a $(2, d)$-sigraph (G, ℓ), the following statements are equivalent:*
(i) (G, ℓ) is ude-balanced.
(ii) There exist two disjoint subsets V_1 and V_2 of $V(G)$, one of them possibly empty, such that $V = V_1 \cup V_2$,

(a) any edge labeled $++$ joins two vertices in V_1, and any edge labeled $--$ joins two vertices in V_2, and

(b) any edge labeled $+-$ is a directionally labeled edge going from V_1 to V_2, and any edge labeled $-+$ is a directionally labeled edge going from V_2 to V_1.

Proof. **(i) $\Longrightarrow$ (ii):** Let (G, ℓ) be ude-balanced. Then for each vertex u in G, either $\ell_1(u) = +$ or $\ell_1(u) = -$. If the label on each edge is $++$ or $--$ then ℓ is trivial and we have nothing to show since V and $\emptyset$ form the required partition. So,we partition the vertex set V into sets V_1 and V_2 as follows:

$$V_1 = \{u \in V : \ell_1(u) = +\},$$

and

$$V_2 = \{v \in V : \ell_1(v) = -\}.$$

Now, suppose uu_1 is an edge labeled $++$. Then, since (G, ℓ) is ude-balanced, we have $\ell_1(u_1) = \ell(u) = +$. This implies, both u and u_1 belong to V_1. Similarly, if vv_1 is an edge labeled $--$, then both v and v_1 belong to V_2.

Suppose now uv is an edge with $\ell(uv) = +-$. Then, $\ell_1(u) = +$ and $\ell_1(v) = -$. This implies $u \in V_1$ and $v \in V_2$. Similarly, if uv is an edge with $\ell(uv) = -+$, then $u \in V_2$, and $v \in V_1$. This proves (ii).

(ii) $\Longrightarrow$ (i): Clearly, (ii) implies that for each vertex u in G either $\ell_1(u) = +$ or $\ell_1(u) = -$, and hence (G, ℓ) is ude-balanced. □

3 Directional adjacency matrix

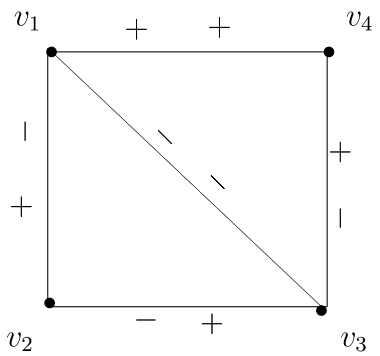
We now show that a $(2, d)$-sigraph can be uniquely represented by a matrix called *directional adjacency matrix*.

This enables one to represent a $(2, d)$-sigraph by a $(-1, 0, 1)$-matrix $M^\ell = [a^\ell_{ij}]$ defined as follows.

A $(-1, 0, 1)$-matrix $M^\ell = [a^\ell_{ij}]$ of order p is the directional adjacency matrix of a $(2, d)$-sigraph (G, ℓ) of order p if

$$a^\ell_{ij} = \begin{cases} 1, \text{when } v_iv_j \text{ is an edge and } \ell(v_iv_j) \in \{++, +-\} \\ -1, \text{when } v_iv_j \text{ is an edge and } \ell(v_iv_j) \in \{--, -+\} \\ 0, \text{otherwise.} \end{cases}$$

A $(2, d)$-sigraph (G, ℓ) can be uniquely represented by its directional adjacency matrix M^ℓ as defined above.



	v_1	v_2	v_3	v_4
v_1	0	−1	−1	1
v_2	1	0	−1	0
v_3	−1	1	0	−1
v_4	1	0	1	0

G : M :

Directional adjacency matrix of a $(2, d)$-sigraph.

Figure 2

In Figure 2, the matrix M^ℓ is the directional adjacency matrix of the $(2, d)$-sigraph G.

Problem 7. *Discuss the spectrum and energy of a $(2, d)$-sigraph using this matrix, where, in* *general, the* energy *of a square real matrix is defined as the sum of the modulii of its eigenvalues.*

Problem 8. *Is there any spectral criterion for a* ude*-balanced* $(2, d)$*-sigraph?*

The following is a characterization of directional adjacency matrix of a $(2, d)$-sigraph G.

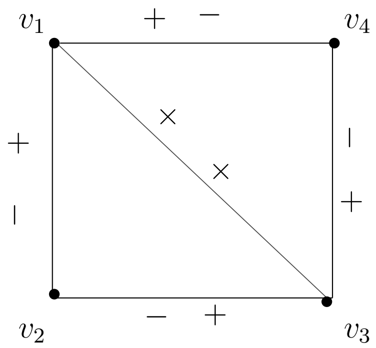
Proposition 9. *A square* $(-1, 0, 1)$*-matrix* $M = [a_{ij}]$ *of order* p *with zero diagonal is the directional adjacency matrix of a* $(2, d)$*-sigraph of order* p *if, and only if,*
(i) $a_{ij} \in \{0, 1, -1\}$
(ii) $|a_{ij}| = |a_{ji}|$.

Proof. ***Necessity:*** This is obvious since the adjacency matrix of any $(2, d)$-sigraph satisfies the conditions mentioned above.
Sufficiency: Let $M = [a_{ij}]$ be a $(-1, 0, 1)$-matrix of order p satisfying the conditions (i) and (ii) above. We construct a $(2, d)$-sigraph (G, ℓ) with vertex set $V = \{v_1, v_2, \ldots, v_p\}$ as follows: In G, v_iv_j is an edge if, and only if, $a_{ij} \neq 0$. If v_iv_j is an edge, the label $\ell(v_iv_j)$ is determined as follows.

$$\ell(v_iv_j) = \begin{cases} ++, \text{if } a_{ij} = a_{ji} = 1 \\ +-, \text{if } a_{ij} = 1, a_{ji} = -1 \\ -+, \text{if } a_{ij} = -1, a_{ji} = 1 \\ --, \text{if } a_{ij} = a_{ji} = -1. \end{cases}$$

The matrix M is then the directional adjacency matrix of the $(2, d)$-sigraph (G, ℓ) thus constructed. □

Definition 10. A $(-1, 0, 1)$-matrix M is *row-balanced* if all the non zero entries in any row are either 1 or -1.



G

	v_1	v_2	v_3	v_4
v_1	0	1	1	1
v_2	-1	0	-1	0
v_3	1	1	0	1
v_4	-1	0	-1	0

M

Directional adjacency matrix of the ude-balanced $(2, d)$-sigraph G.
Figure 3

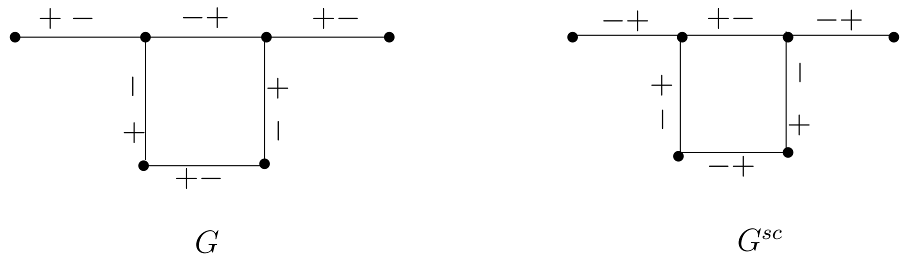
The following is a characterization of a ude-balanced $(2, d)$-sigraph in terms of its directional adjacency matrix.

Proposition 11. *For a* $(2, d)$*-sigraph* (G, ℓ) *the following statements are equivalent.*
(i) G is ude*-balanced.*
(ii) The directional adjacency matrix $M^\ell = [a^\ell_{ij}]$ *is row-balanced.*

Proof. Let ℓ be a ude-labeling of the graph $G = (V, E)$ with $V = \{v_1, v_2, \ldots, v_p\}$.
G is ude-balanced
$\Longleftrightarrow \ell_1(v_i) = 1$ or -1, for $1 \leq i \leq p$
$\Longleftrightarrow$ each non-zero entry in the i^{th} row of M^ℓ is 1 or -1
$\Longleftrightarrow M^\ell$ is row-balanced. □

4 s-Complement of a $(2, d)$-sigraph

The *s-complement* G^{sc} of a $(2, d)$-sigraph (G, ℓ^{sc}) is the $(2, d)$-sigraph obtained from (G, ℓ) by interchanging the signs $+$ and $-$ in ℓ; ℓ^{sc} is then called the *s-complement* of ℓ.



$(2, d)$-sigraph G and its s-complement.
Figure 4

In Figure 4, G^{sc} is the s-complement of G.
One can easily see the validity the following fact.

Proposition 12. *A $(2, d)$-sigraph $G = (V, E)$ is ude-balanced if, and only if, its s-complement $G^{(sc)}$ is ude-balanced.*

For example, in Figure 4, both G and G^{sc} are ude-balanced.

5 Induced sigraph of a $(2, d)$-sigraph

The *induced sigraph* (G, σ^ℓ) of a $(2, d)$-sigraph (G, ℓ) is the sigraph obtained by assigning to each edge uv of G the product of the signs in $\ell(uv)$.

The following result relates ude-balanced $(2, d)$-sigraph (G, ℓ) with the sigraph (G, σ^ℓ).

Proposition 13. *If G is ude-balanced $(2, d)$-sigraph (G, ℓ), then the sigraph (G, σ^ℓ) is balanced. But, the converse is not true.*

Proof. If G is a ude-balanced $(2, d)$-sigraph (G, ℓ), then by Proposition 6, there exists a partition $\{V_1, V_2\}$ of $V(G)$ such that every edge in G labeled $++$ joins two vertices in V_1, every edge labeled $--$ joins two vertices in V_2, and every edge uv directionally labeled $+-$ joins a vertex u in V_1 and a vertex v in V_2. This implies, in the induced sigraph (G, σ^ℓ), this partition is such that every positive edge joins two vertices in V_1 or in V_2, and every negative edge joins a vertex of V_1 and a vertex of V_2. Hence, by Proposition 1, (G, σ^ℓ) is balanced. □

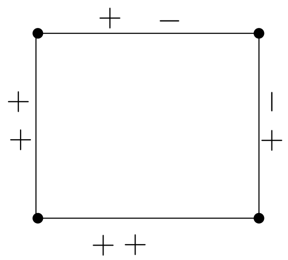
The converse is not true. For example, in Figure 5 the induced sigraph (G, σ^ℓ) is balanced though (G, ℓ) is not ude-balanced.

Corollary 14. *If (G, ℓ) is ude-balanced $(2, d)$-sigraph, then every cycle in the $(2, d)$-sigraph (G, ℓ) has an even number of edges directionally labeled by the ordered pairs in $\{+-, -+\}$.*

Proof. Let (G, ℓ) be a ude-balanced $(2, d)$-sigraph. Then, by Proposition 13 (G, σ^ℓ) is balanced. Hence, every cycle in (G, σ^ℓ) has an even number of negative edges. This implies, every cycle in (G, σ^ℓ) has an even number of edges directionally labeled by the ordered pairs in $\{+-, -+\}$. $\square$

Note that the converse of Corollary 14 is not true. For example, in Figure 5, the cycle has an even number of edges that are directionally labeled by the ordered pairs in $\{+-, -+\}$. But, the cycle is not ude-balanced.

$G:$



A $(2, d)$-sigraph which is not ude-balanced.

Figure 5

Remark: Given a balanced sigraph (G, σ) one can construct a ude-balanced $(2, d)$-sigraph (G, ℓ) such that (G, σ^ℓ) is the induced sigraph of (G, ℓ) as follows: Since (G, ℓ) is ude-balanced, there exists a partition $\{V_1, V_2\}$ of $V(G)$ satisfying the conditions of Proposition 1.

If there is a positive edge in V_1, label it by $++$. If there is a positive edge in V_2, label it by $--$. Also, directionally label each edge going from V_1 to V_2 by $+-$. Then, the resulting $(2, d)$-sigraph is ude-balanced.

6 Splitting of a $(2, d)$-sigraph (G, ℓ) into two sigraphs G_1 and G_2

Let V be the vertex set of a $(2, d)$-sigraph (G, ℓ). We obtain two sigraphs G_1 and G_2 having the same vertex set of G as follows.

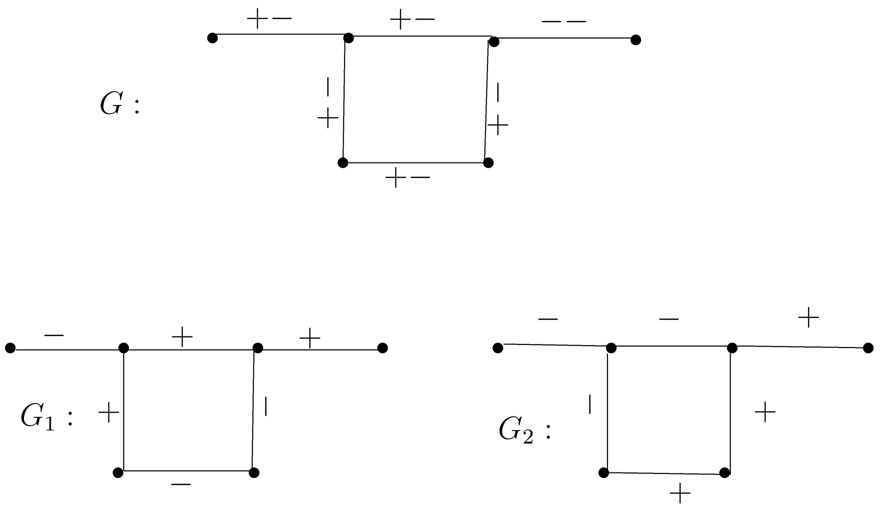
If uv is an edge in G, then uv is an edge both G_1 and G_2. Further, if an edge uv in G does not lie on a cycle in G, then the sign of the edge uv in both G_1 and G_2 is equal to the product of the signs on uv in G.

If an edge in G belongs to a cycle, then the signs of the corresponding edges in the sigraphs G_1 and G_2 are determined as follows:

Let $C : v_1v_2 \dots v_nv_1$ be a cycle in (G, ℓ), and $a_ib_i \in \{++, --, +-, -+\}$, $1 \leq i \leq n$. Suppose in (G, ℓ), $\ell(v_iv_{i+1}) = a_ib_i$, $1 \leq i \leq n-1$, and $\ell(v_nv_1) = a_nb_n$.

In G_1, the sign on the edge v_iv_{i+1} is a_i, $1 \le i \le n-1$, and the sign on the edge v_nv_1 is a_n. In G_2 , the sign on the edge v_iv_{i+1} is b_i, $1 \le i \le n-1$, and the sign on v_nv_1 is b_n. We say that the two sigraphs G_1 and G_2 are obtained by a 2-*splitting* of the $(2, d)$-sigraph (G, ℓ).
Note: The two sigraphs G_1 and G_2 obtained by a 2-splitting of a $(2, d)$ signed graph (G, ℓ) are not unique.

Definition 15. A $(2, d)$-sigraph (G, ℓ) is 2-*balanced* if both the sigraphs G_1 and G_2 obtained by a 2-splitting of (G, ℓ) are balanced.



G_1 and G_2 are obtained by a 2-splitting of G

Figure 6

Note that if the $(2, d)$-sigraph (G, ℓ) has no cycles, then (G, ℓ) is 2-balanced, since then the two sigraphs G_1 and G_2 obtained by a 2-splitting (G, ℓ) are balanced. In fact, in this case G_1 and G_2 are identical.

We now obtain a characterization of 2-balanced $(2, d)$-sigraph (G, ℓ) similar to Proposition 2.

Theorem 16. *(Characterization) A $(2, d)$-sigraph (G, ℓ) is 2-balanced if, and only if, there exists a marking of its vertices by ordered pairs $a_ib_i \in \{++, --, +-, -+\}$ such that for any edge uv in (G, ℓ), $\ell(uv)$ is the product of the markings of u and v.*

Proof. Let G_1 and G_2 be two sigraphs obtained by a 2-splitting of a $(2, d)$-sigraph (G, ℓ). Then both G_1 and G_2 are balanced sigraphs. By Proposition 2, there exist a marking say m_1 in G_1 and m_2 in G_2 such that for any edge uv in G, the sign of uv is $m_1(u)m_1(v) = a_{i1}a_{i2}$, where $a_{i1} = m_1(u)$ and $a_{i2} = m_1(v)$. Similarly in G_2, the sign of uv is $m_2(u).m_2(v) = b_{i1}b_{i2}$, where $b_{i1} = m_2(u)$ and $b_{i2} = m_2(v)$. Let the label $\ell(uv)$ on the edge uv in $(G, \ell) = a_ib_i$. Then the sign of the edge uv in G_1 is a_i, and the sign of uv in G_2 is b_i. Hence, $a_i = a_{i1}a_{i2}$, and $b_i = b_{i1}b_{i2}$. If we assign the markings $a_{i1}b_{i1}$ to u, $a_{i2}b_{i2}$ to v in (G, ℓ), we find that the label a_ib_i on uv in (G, ℓ) is the product

$$(a_{i1}b_{i1})(a_{i2}b_{i2}) = (a_{i1}a_{i2}b_{i1}b_{i2}) = a_ib_i.$$

The converse follows by retracing the steps above. □

Note:
1) A ude-balanced $(2, d)$-sigraph need not be 2-balanced.
For example, define ℓ on $C_5 = (u_1, u_2, u_3, u_4, u_5, u_1)$ by letting

$$\ell(u_1u_2) = --, \ell(u_2u_3) = -+, \ell(u_3u_4) = +-, \ell(u_4u_5) = -+, \ell(u_5u_1) = +-.$$

Then, it is easy to verify that C_5 is ude-balanced, but it is not 2-balanced.

2) A 2-balanced $(2, d)$-sigraph need not be ude-balanced.
For example, in Figure 5 the $(2, d)$-sigraph is 2-balanced, but not ude-balanced.

7 Clusterable $(2, d)$-sigraphs

A signed graph (G, σ) is *clusterable* if its vertex set can be partitioned into subsets $V_1, V_2, \ldots, V_k$, such that every positive edge joins two vertices in V_i, $1 \leq i \leq k$, and every negative edge joins two vertices in different sets of the partition. The following is a well known theorem of Davis [1].

Theorem 17. *([1]) A signed graph is clusterable if, and only if, it has no cycle with exactly one negative edge.*

Analogously, one can define clusterability in a $(2, d)$-sigraph as follows:

Definition 18. A $(2, d)$-sigraph (G, ℓ) is *clusterable* if $V(G)$ can be partitioned into subsets $V_1, V_2, \ldots, V_k$ such that every edge having a label in $\{+-, -+\}$ joins two vertices in different sets of the above partition, and every edge with labels in $\{--, ++\}$ joins two vertices of only one of the subsets V_i, $1 \leq i \leq k$.

For example, the $(2, d)$-sigraph G in Figure 5 is clusterable. The following fact is easy to see by observing the nature of the product rule that is used to define the induced sigraph of a $(2, d)$-sigraph.

Lemma 19. *A $(2, d)$-sigraph (G, ℓ) is clusterable if, and only if, its induced sigraph (G, σ^ℓ) is clusterable.*

The following is a characterization of clusterable $(2, d)$-sigraphs.

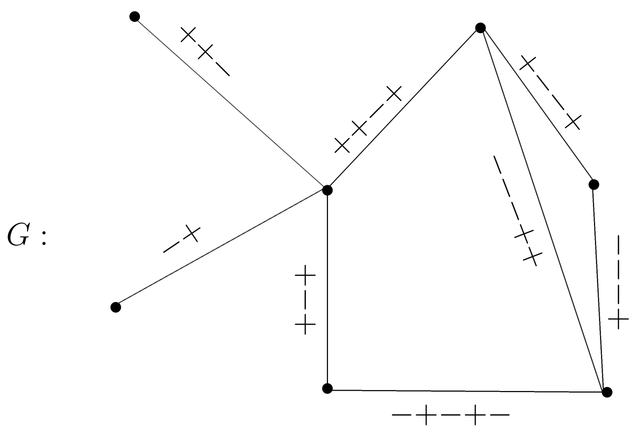
Proposition 20. *A $(2, d)$-sigraph (G, ℓ) is clusterable if, and only if, it has no cycle with exactly one edge having a label belonging to $\{+-, -+\}$.*

Proof. This follows directly from Theorem 17 and Lemma 19. □

Note: A $(2, d)$-sigraph is clusterable if, and only if, its negation is clusterable. But this is not true in the case of sigraphs. If a sigraph G is clusterable then its negation need not be clusterable.

8 (n, d)-Sigraphs

A (n, d)-sigraph is a graph $G = (V, E)$ in which every edge uv is directionally labeled by an n-tuple $(a_1, a_2, \ldots, a_n)$, for various integers $n \geq 2$, and $a_i \in \{+, -\}$, $1 \leq i \leq n$ such that the label on the edge uv in the direction from u to v is $\ell(uv) = (a_1, a_2, \ldots, a_n)$ and $\ell(vu) = (a_n, a_{n-1}, \ldots, a_1)$. For example, in Figure 7, G is directionally labeled n-tuple signed graph.



Directionally labeled n-tuple signed graph
Figure 7

A (n, d)-sigraph is n-*uniform* if each edge is directionally labeled by an n-tuple for some fixed positive integer $n \geq 2$. For details on n-uniform (n, d)-sigraphs, see ([6, 7]). In [8, 9], some applications of (n, d)-sigraphs are given when $n = 3, 4$.

We now extend the concepts of uniform directional labeling, balance and clusterability of $(2, d)$-sigraphs to (n, d)-sigraphs.

9 Induced $(2, d)$-sigraph G_2 of an (n, d)-sigraph G

Let $G = (V, E)$ be a (n, d)-sigraph. The *induced* $(2, d)$-sigraph of G, denoted by G_2 has the same vertex set, and edge set as G, where the directional labeling of the edges are defined as follows: Suppose for an edge uv in G, $\ell(uv) = (a_1, a_2, \ldots, a_n)$, $n \geq 2$. For a given positive integer $n \geq 3$, we define the products a and b as follows: Let n be even, and $r = \frac{n}{2}$. Then $a = \prod_{i=1}^{r} a_i$, $b = \prod_{j=r+1}^{n} a_j$ where a_i, $a_j \in \{+, -\}$. Let n be an odd integer, and $r = \lceil \frac{n}{2} \rceil$. Then $a = \prod_{i=1}^{r} a_i$, $b = \prod_{j=r}^{n} a_j$. Then corresponding to the label $\ell(uv) = (a_1, a_2, \ldots, a_n)$ on the edge uv in G, we define the label on the edge uv in G_2 as $\ell(uv) = ab$. In the Figure 8, G_2 is the induced $(2, d)$-sigraph of the (n, d)-sigraph G.

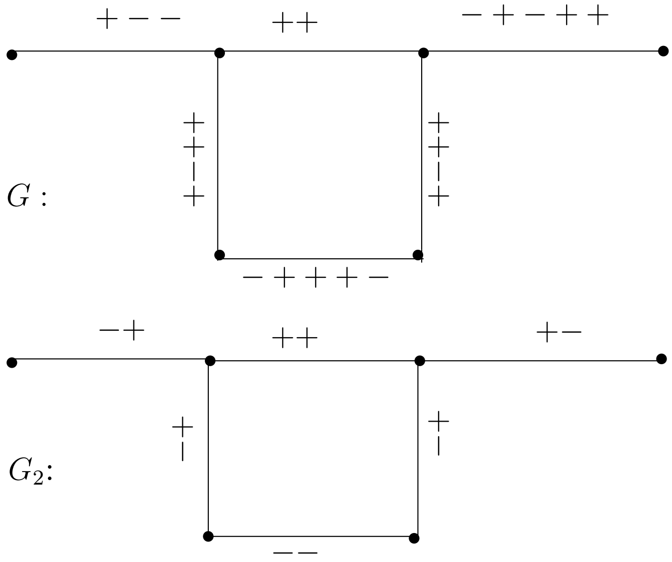


Figure 8

Definition 21. *Let G be a (n, d)-sigraph. Then,*
(i) G has a ude-labeling at a vertex u if there is such a labeling at u in its induced $(2, d)$-sigraph G_2,
(ii) G has ude-labeling if it has such a labeling at all its vertices, and
(iii) G is ude-balanced if its induced $(2, d)$-sigraph G_2 is ude-balanced.

As a direct consequence of Proposition 6, we have the following characterization of ude-balanced (n, d)-sigraphs.

Proposition 22. *For an (n, d)-sigraph $G = (V, E)$, the following statements are equivalent.*
(i) G is ude-balanced.
(ii) There exists a partition $V = V_1 \cup V_2$ of vertex set V of G such that in G_2,
(a) any edge labeled $++$ joins two vertices in V_1 and any edge labeled $--$ joins two vertices in V_2,
(b) any edge labeled $+-$ is a directionally labeled edge going from V_1 to V_2.

For example, in Figure 8, the (n, d) sigraph is ude-balanced since its induced $(2, d)$-sigraph G_2 is ude-balanced.

10 Clusterable (n, d)-sigraphs

Definition 23. *A (n, d)-sigraph G is* clusterable *if its induced $(2, d)$-sigraph G_2 is clusterable.*

With suitable changes, Proposition 20 gives a characterization of clusterable (n, d)-sigraphs.

11 $(2, d)$-Sigraphs and bidirected graphs

A *bidirected graph* $B = (G, \beta)$ is a graph $G = (V, E)$ in which each end u of every edge e receives a label $\beta(u, e) \in \{+, -\}$; G is called the *underlying graph* of (G, β) and β is called a *bidirection* of G. In particular, if $\beta(u, e) = +$ then it denotes an arrow on e pointed into the vertex u and if $\beta(u, e) = -$ then it denotes an arrow on e directed out of u. Thus, in a bidirected graph each end of an edge has an independent direction. Bidirected graphs were defined by Edmonds [2]. There is a close connection between $(2, d)$-sigraphs and bidirected graphs. For details see [11].

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