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A note on the modified q-Dedekind sums

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Abstract: In the present paper, the fundamental aim is to consider a p-adic continuous function for an odd prime to inside a p-adic q-analogue of the higher order Extended Dedekind-type sums related to q-Genocchi polynomials with weight α by using fermionic p-adic invariant q-integral on $\mathbb{Z}_p$.

Keywords: Dedekind Sums, q-Dedekind-type Sums, p-adic q-integral, q-Genocchi polynomials with weight α.
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1 Introduction

Imagine that p be a fixed odd prime number. We now start with definition of the following notations. Let $\mathbb{Q}_p$ be the field p-adic rational numbers and let $\mathbb{C}_p$ be the completion of algebraic closure of $\mathbb{Q}_p$.

Thus,

$$\mathbb{Q}_p = \left\{ x = \sum_{n=-k}^{\infty} a_n p^n : 0 \leq a_n < p \right\}.$$

Then $\mathbb{Z}_p$ is integral domain, which is defined by

$$\mathbb{Z}_p = \left\{ x = \sum_{n=0}^{\infty} a_n p^n : 0 \leq a_n < p \right\}$$

or

$$\mathbb{Z}_p = \left\{ x \in \mathbb{Q}_p : |x|_p \leq 1 \right\}.$$

We assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$ as an indeterminate. The p-adic absolute value $|.|_p$, is normally defined by

$$|x|_p = \frac{1}{p^n}$$

where $x = p^n \frac{s}{t}$ with $(p, s) = (p, t) = (s, t) = 1$ and $n \in \mathbb{Q}$ (for details, see [1-19]).

The p-adic q-Haar distribution is defined by Kim as follows: for any postive integer n,

$$\mu_q \left(a + p^n \mathbb{Z}_p \right) = (-q)^a \frac{(1+q)}{1+q^{p^n}}$$

for $0 \leq a < p^n$ and this can be extended to a measure on $\mathbb{Z}_p$ (for details, see [12], [14], [17]).

In [7], the q-Genocchi polynomials are defined by Araci *et al.* as follows:

$$\widetilde{G}_{n,q}^{(\alpha)}(x) = n \int_{\mathbb{Z}_p} \left(\frac{1-q^{\alpha(x+\xi)}}{1-q^{\alpha}} \right)^{n-1} d\mu_q(\xi) \tag{1}$$

for $n \in \mathbb{Z}_+ := \{0, 1, 2, 3, \cdots\}$. We easily see that

$$\lim_{q \to 1} \widetilde{G}_{n,q}^{(\alpha)}(x) = G_n(x)$$

where $G_n(x)$ are Genocchi polynomials, which are given in the form:

$$\sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} = e^{tx} \frac{2t}{e^t+1}, \ \ |t| < \pi$$

(for details, see [7]). Taking $x = 0$ into (1), then we have $\widetilde{G}_{n,q}^{(\alpha)}(0) := \widetilde{G}_{n,q}^{(\alpha)}$ are called q-Genocchi numbers with weight α.

The q-Genocchi numbers and polynomials have the following identities:

$$\widetilde{G}_{n+1,q}^{(\alpha)} = (n+1) \frac{1+q}{(1-q^{\alpha})^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+q^{\alpha l+1}}, \tag{2}$$

$$\widetilde{G}_{n+1,q}^{(\alpha)}(x) = (n+1) \frac{1+q}{(1-q^{\alpha})^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{q^{\alpha l x}}{1+q^{\alpha l+1}}, \tag{3}$$

$$\widetilde{G}_{n,q}^{(\alpha)}(x) = \sum_{l=0}^{n} \binom{n}{l} q^{\alpha l x} \widetilde{G}_{l,q}^{(\alpha)} \left(\frac{1-q^{\alpha x}}{1-q^{\alpha}} \right)^{n-l}. \tag{4}$$

Additionally, for d odd natural number, we have

$$\widetilde{G}_{n,q}^{(\alpha)}(dx) = \left(\frac{1+q}{1+q^d} \right) \left(\frac{1-q^{\alpha d}}{1-q^{\alpha}} \right)^{n-1} \sum_{a=0}^{d-1} q^a (-1)^a \widetilde{G}_{n,q}^{(\alpha)} \left(x + \frac{a}{d} \right), \tag{5}$$

(for details about this subject, see [7]).

For any positive integer h, k and m, Dedekind-type DC sums are given by Kim in [9], [10] and [11] as follows:

$$S_m(h,k)=\sum_{M=1}^{k-1}(-1)^{M-1}\frac{M}{k}\overline{E}_m\left(\frac{hM}{k}\right)$$

where $\overline{E}_m(x)$ are the m-th periodic Euler function.

In 2011, Taekyun Kim added a weight to q-Bernoulli polynomials in [16]. He derived not only new but also interesting properties for weighted q-Bernoulli polynomials. After, many mathematicians, by utilizing from Kim's paper [16], have introduced a new concept in Analytic numbers theory as weighted q-Bernoulli, weighted q-Euler, weighted q-Genocchi polynomials in [17], [6], [7], [1], [3] and [5]. Also, the generating function of weighted q-Genocchi polynomials was introduced by Araci *et al.* in [7]. They also derived several arithmetic properties for weighted q-Genocchi polynomials.

Kim has given some interesting properties for Dedekind-type DC sums. He firstly considered a p-adic continuous function for an odd prime number to contain a p-adic q-analogue of the higher order Dedekind-type DC sums $k^m S_{m+1}(h,k)$ in [10].

By the same motivation, we, by using p-adic invariant q-integral on $\mathbb{Z}_p$, shall get weighted p-adic q-analogue of the higher order Dedekind-type DC sums $k^m S_{m+1}(h,k)$.

2 Extended q-Dedekind-type sums in connection with q-Genocchi polynomials with weight α

If x is a p-adic integer, then $w(x)$ is the unique solution of $w(x)=w(x)^p$ that is congruent to x mod p. It can also be defined by

$$w(x)=\lim_{n\to\infty}x^{p^n}.$$

The multiplicative group of p-adic units is a product of the finite group of roots of unity, and a group isomorphic to the p-adic integers. The finite group is cylic of order $p-1$ or 2, as p is odd or even, respectively, and so it is isomorphic. Actually, the teichmüller character gives a canonical isomorphism between these two groups.

Let w be the $Teichm\ddot{u}ller$ character (mod p). For $x\in\mathbb{Z}_p^*:=\mathbb{Z}_p/p\mathbb{Z}_p$, set

$$\langle x:q\rangle=w^{-1}(x)\left(\frac{1-q^x}{1-q}\right).$$

Let a and N be positive integers with $(p,a)=1$ and $p\mid N$. We now introduce the following

$$\widetilde{E}_q^{(\alpha)}\left(s,a,N:q^N\right)=w^{-1}(a)\langle x:q^\alpha\rangle^s\sum_{j=0}^{\infty}\binom{s}{j}q^{\alpha aj}\left(\frac{1-q^{\alpha N}}{1-q^{\alpha a}}\right)^j\widetilde{G}_{j,q^N}^{(\alpha)}.$$

In particular, if $m+1\equiv 0(\mathrm{mod}p-1)$, then we have

$$\begin{aligned}\widetilde{E}_q^{(\alpha)}\left(m,a,N:q^N\right) &= \left(\frac{1-q^{\alpha a}}{1-q^\alpha}\right)^m\sum_{j=0}^{m}\binom{m}{j}q^{\alpha aj}\widetilde{G}_{j,q^N}^{(\alpha)}\left(\frac{1-q^{\alpha N}}{1-q^{\alpha a}}\right)^j\\ &= \left(\frac{1-q^{\alpha N}}{1-q^\alpha}\right)^m\int_{\mathbb{Z}_p}\left(\frac{1-q^{\alpha N\left(\xi+\frac{a}{N}\right)}}{1-q^{\alpha N}}\right)^m d\mu_{q^N}(\xi).\end{aligned}$$

Then, $\widetilde{E}_q^{(\alpha)}\left(m, a, N : q^N\right)$ is a continuous p-adic extension of

$$\left(\frac{1-q^{\alpha N}}{1-q^{\alpha}}\right)^m \frac{\widetilde{G}_{m+1,q^N}^{(\alpha)}\left(\frac{a}{N}\right)}{m+1}.$$

Suppose that $[.]$ be the Gauss' symbol and let $\{x\} = x - [x]$. Thus, we are now ready to treat q-extension of the higher order Dedekind-type DC sums $\widetilde{S}_{m,q}^{(\alpha)}\left(h, k : q^l\right)$ in the form:

$$\widetilde{S}_{m,q}^{(\alpha)}\left(h, k : q^l\right) = \sum_{M=1}^{k-1} (-1)^{M-1} \left(\frac{1-q^{\alpha M}}{1-q^{\alpha k}}\right) \int_{\mathbb{Z}_p} \left(\frac{1-q^{\alpha\left(l\xi + l\left\{\frac{hM}{k}\right\}\right)}}{1-q^{\alpha l}}\right)^m d\mu_{q^l}(\xi).$$

If $m+1 \equiv 0 \ (\mathrm{mod}\ p-1)$,

$$\left(\frac{1-q^{\alpha k}}{1-q^{\alpha}}\right)^{m+1} \sum_{M=1}^{k-1} (-1)^{M-1} \left(\frac{1-q^{\alpha M}}{1-q^{\alpha k}}\right) \int_{\mathbb{Z}_p} \left(\frac{1-q^{\alpha k\left(\xi + \frac{hM}{k}\right)}}{1-q^{\alpha k}}\right)^m d\mu_{q^k}(\xi)$$
$$= \sum_{M=1}^{k-1} (-1)^{M-1} \left(\frac{1-q^{\alpha M}}{1-q^{\alpha}}\right) \left(\frac{1-q^{\alpha k}}{1-q^{\alpha}}\right)^m \int_{\mathbb{Z}_p} \left(\frac{1-q^{\alpha k\left(\xi + \frac{hM}{k}\right)}}{1-q^{\alpha k}}\right)^m d\mu_{q^k}(\xi)$$

where $p \mid k$, $(hM, p) = 1$ for each M. Via the equation (1), we easily state the following

$$\left(\frac{1-q^{\alpha k}}{1-q^{\alpha}}\right)^{m+1} \widetilde{S}_{m,q}^{(\alpha)}\left(h, k : q^k\right) \tag{6}$$
$$= \sum_{M=1}^{k-1} \left(\frac{1-q^{\alpha M}}{1-q^{\alpha}}\right) \left(\frac{1-q^{\alpha k}}{1-q^{\alpha}}\right)^m (-1)^{M-1} \int_{\mathbb{Z}_p} \left(\frac{1-q^{\alpha k\left(\xi + \frac{hM}{k}\right)}}{1-q^{\alpha k}}\right)^m d\mu_{q^k}(\xi)$$
$$= \sum_{M=1}^{k-1} (-1)^{M-1} \left(\frac{1-q^{\alpha M}}{1-q^{\alpha}}\right) \widetilde{E}_q^{(\alpha)}\left(m, (hM)_k : q^k\right)$$

where $(hM)_k$ denotes the integer x such that $0 \leq x < n$ and $x \equiv \alpha \ (\mathrm{mod}\ k)$.

It is simple to show the following:

$$\int_{\mathbb{Z}_p} \left(\frac{1-q^{\alpha(x+\xi)}}{1-q^{\alpha}}\right)^k d\mu_q(\xi) \tag{7}$$
$$= \left(\frac{1-q^{\alpha m}}{1-q^{\alpha}}\right)^k \frac{1+q}{1+q^m} \sum_{i=0}^{m-1} (-1)^i \int_{\mathbb{Z}_p} \left(\frac{1-q^{\alpha m\left(\xi + \frac{x+i}{m}\right)}}{1-q^{\alpha m}}\right)^k d\mu_{q^m}(\xi).$$

Due to (6) and (7), we easily obtain

$$\left(\frac{1-q^{\alpha N}}{1-q^{\alpha}}\right)^m \int_{\mathbb{Z}_p} \left(\frac{1-q^{\alpha N\left(\xi + \frac{a}{N}\right)}}{1-q^{\alpha N}}\right)^m d\mu_{q^N}(\xi) \tag{8}$$
$$= \frac{1+q^N}{1+q^{Np}} \sum_{i=0}^{p-1} (-1)^i \left(\frac{1-q^{\alpha Np}}{1-q^{\alpha}}\right)^m \int_{\mathbb{Z}_p} \left(\frac{1-q^{\alpha pN\left(\xi + \frac{a+iN}{pN}\right)}}{1-q^{\alpha pN}}\right)^m d\mu_{q^{pN}}(\xi).$$

Thanks to (6), (7) and (8), we discover the following p-adic integration:

$$\widetilde{E}_q^{(\alpha)}\left(s, a, N : q^N\right) = \frac{1+q^N}{1+q^{Np}} \sum_{\substack{0 \leq i \leq p-1 \\ a+iN \neq 0 (\mathrm{mod} p)}} (-1)^i \, \widetilde{E}_q^{(\alpha)}\left(s, (a+iN)_{pN}, p^N : q^{pN}\right).$$

On the other hand,

$$\widetilde{E}_q^{(\alpha)}\left(m,a,N:q^N\right)=\left(\frac{1-q^{\alpha N}}{1-q^{\alpha}}\right)^m\int_{\mathbb{Z}_p}\left(\frac{1-q^{\alpha N\left(\xi+\frac{a}{N}\right)}}{1-q^{\alpha N}}\right)^m d\mu_{q^N}(\xi)$$
$$-\left(\frac{1-q^{\alpha Np}}{1-q^{\alpha}}\right)^m\int_{\mathbb{Z}_p}\left(\frac{1-q^{\alpha pN\left(\xi+\frac{a+iN}{pN}\right)}}{1-q^{\alpha pN}}\right)^m d\mu_{q^{pN}}(\xi)$$

where $(p^{-1}a)_N$ denotes the integer x with $0\leq x<N$, $px\equiv a$ (mod N) and m is integer with $m+1\equiv 0$ (mod $p-1$). Therefore, we can state the following

$$\sum_{M=1}^{k-1}(-1)^{M-1}\left(\frac{1-q^{\alpha M}}{1-q^{\alpha}}\right)\widetilde{E}_q^{(\alpha)}\left(m,hM,k:q^k\right)$$
$$=\left(\frac{1-q^{\alpha k}}{1-q^{\alpha}}\right)^{m+1}\widetilde{S}_{m,q}^{(\alpha)}\left(h,k:q^k\right)-\left(\frac{1-q^{\alpha k}}{1-q^{\alpha}}\right)^{m+1}\left(\frac{1-q^{\alpha kp}}{1-q^{\alpha k}}\right)\widetilde{S}_{m,q}^{(\alpha)}\left(\left(p^{-1}h\right),k:q^{pk}\right)$$

where $p\nmid k$ and $p\nmid hm$ for each M. Thus, we obtain the following definition, which seems interesting for further studying in theory of Dedekind sums.

Definition 2.1 *Let h,k be positive integers with $(h,k)=1$, $p\nmid k$. For $s\in\mathbb{Z}_p$, we define p-adic Dedekind-type DC sums as follows:*

$$\widetilde{S}_{p,q}^{(\alpha)}\left(s:h,k:q^k\right)=\sum_{M=1}^{k-1}(-1)^{M-1}\left(\frac{1-q^{\alpha M}}{1-q^{\alpha}}\right)\widetilde{E}_q^{(\alpha)}\left(m,hM,k:q^k\right).$$

As a result of the above definition, we derive the following theorem.

Theorem 2.2 *For $m+1\equiv 0$ (mod $p-1$) and $(p^{-1}a)_N$ denotes the integer x with $0\leq x<N$, $px\equiv a$ (mod N), then, we have*

$$\widetilde{S}_{p,q}^{(\alpha)}\left(s:h,k:q^k\right)=\left(\frac{1-q^{\alpha k}}{1-q^{\alpha}}\right)^{m+1}\widetilde{S}_{m,q}^{(\alpha)}\left(h,k:q^k\right)$$
$$-\left(\frac{1-q^{\alpha k}}{1-q^{\alpha}}\right)^{m+1}\left(\frac{1-q^{\alpha kp}}{1-q^{\alpha k}}\right)\widetilde{S}_{m,q}^{(\alpha)}\left(\left(p^{-1}h\right),k:q^{pk}\right).$$

In the special case $\alpha=1$, our applications in theory of Dedekind sums resemble Kim's results in [10]. These results seem to be interesting for further studies in [9], [11] and [18].

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