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On subsets of finite Abelian groups without non-trivial solutions of $x_1 + x_2 + \cdots + x_s - sx_{s+1} = 0$

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Abstract: Let $D(G)$ be the maximal cardinality of a set $A \subseteq G$ that contains no non-trivial solution to $x_1 + \cdots + x_s - sx_{s+1} = 0$ with $x_i \in A\,(1 \leq i \leq s+1)$. Let

$$d(n) = \sup_{\mathrm{rk}(H) \geq n} \frac{D(H)}{|H|},$$

where $\mathrm{rk}(H)$ is the rank of H. We prove that for any $n \in \mathbb{N}$, $d(n) \leq \frac{\mathcal{C}}{n^{s-2}}$, where $\mathcal{C}$ is a fixed constant depending only on s.

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1 Introduction

For any natural number $m \geq 3$, let $\mathbf{r} \in (\mathbb{Z} \setminus \{0\})^m$ satisfy $r_1 + \cdots + r_m = 0$. Given a non-trivial finite Abelian group G, we can write $G \simeq \mathbb{Z}_{k_1} \oplus \cdots \oplus \mathbb{Z}_{k_n}$, where $\mathbb{Z}_{k_i}$ is a non-trivial cyclic group of order $k_i\,(1 \leq i \leq n)$ and $k_i | k_{i-1}\,(2 \leq i \leq n)$. We let $\mathrm{rk}(G) = n$ denote the rank of G. A solution $\mathbf{x}$ of $r_1x_1 + \cdots + r_mx_m = 0$ is called *trivial* if $x_i = x_j$ for any $i \neq j$. Otherwise, we say that a solution $\mathbf{x}$ is *non-trivial*. Let $D_{\mathbf{r}}(G)$ be the maximal cardinality of a set $A \subseteq G$ that contains no non-trivial solution to $r_1x_1 + \cdots + r_mx_m = 0$ with $x_i \in A\,(1 \leq i \leq s)$, and write

$$d_{\mathbf{r}}(n) = \sup_{\mathrm{rk}(G) \geq n} \frac{D_{\mathbf{r}}(G)}{|G|}.$$

Note that $D_{(1,1,-2)}(G)$ is the maximum size of a subset $A \subseteq G$ free from non-trivial three-term arithmetic progressions.

Meshulam [3] showed that if $\gcd(|G|, 2) = 1$, then $d_{(1,1,-2)}(G) \leq |G|/\operatorname{rk}(G)$. Lev [1] later established that $d_{(1,1,-2)}(G) \leq 2/\operatorname{rk}(2G)$, where $2G = \{2x : x \in G\}$. Liu and Spencer [2] proved that for any fixed $\mathbf{r} \in (\mathbb{Z} \setminus \{0\})^m$ satisfying $r_1 + \cdots + r_m = 0$, there exists a positive constant $C(\mathbf{r})$ such that whenever $\gcd(|G|, k_1) = 1$, we have $d_{\mathbf{r}}(G) \leq C(\mathbf{r})/(\operatorname{rk}(G))^{m-2}$. In this brief note, we establish a similar theorem without a condition on the $\gcd$ when $\mathbf{r} = (1, 1, \ldots, 1, -s) \in (\mathbb{Z} \setminus \{0\})^{s+1}$ and $s \geq 3$. Namely, we prove the following theorem.

Theorem 1. For $s \geq 3$, $\vec{r} = (1_1, \ldots, 1_s, -s)$, and

$$\mathcal{C} = \max\left\{ \left(\frac{2s-4}{e\log(2)}\right)^{s-2} \sqrt{s^2+s},\ 2(2^{s-1}-2)^{s-2} \right\},$$

we have that for all $n \in \mathbb{N}$, $d(n) \leq \mathcal{C}/n^{s-2}$.

2 Proof of Theorem 1

Let $s \geq 3$ and $\vec{r} = (1, \ldots, 1, -s)$. For a finite Abelian group G, let $\widehat{G}$ denote the character group of G, which is the set of all homomorphisms from G to $\mathbb{C}^\times$. Write χ_0 for the trivial character. For $1 \leq i \leq s+1$, let

$$f_i(\chi) = \sum_{x \in A} \chi(r_i x) = \sum_{x \in A} \chi^{r_i}(x).$$

In what follows, for convenience, we write $D(G)$ in place of $D_{\mathbf{r}}(G)$ and $d(n)$ in place of $d_{\mathbf{r}}(n)$. Before proving Theorem 1, we establish two lemmas necessary for our proof.

Lemma 2. Let G be a finite Abelian group, and suppose that $A \subseteq G$ contains no non-trivial solution to $x_1 + \cdots + x_s - s x_{s+1} = 0$ with $x_i \in A\,(1 \leq i \leq s+1)$. Then,

$$\sum_{\chi \in \widehat{G}} f_1(\chi) f_2(\chi) \cdots f_{s+1}(\chi) \leq |G|\,|A|^{s-1} \binom{s+1}{2}.$$

Proof. We have that

$$\sum_{\chi \in \widehat{G}} f_1(\chi) f_2(\chi) \cdots f_{s+1}(\chi) = \sum_{x_1 \in A} \cdots \sum_{x_{s+1} \in A} \sum_{\chi \in G} \chi(x_1 + x_2 + \cdots + x_s - s x_{s+1}). \tag{1}$$

By [4, Corollary on p. 63],

$$\sum_{\chi \in \widehat{G}} \chi(x) = \begin{cases} |G|, & \text{if } x = 0, \\ 0, & \text{if } x \neq 0. \end{cases}$$

Thus, the sum

$$\sum_{\chi\in\widehat{G}} \chi(x_1+\cdots+x_s-sx_{s+1})$$

detects whether or not $\mathbf{r}\cdot\mathbf{x}=x_1+\cdots+x_s-sx_{s+1}=0$. Since $A\subseteq G$ contains no non-trivial solution to $x_1+\cdots+x_s-sx_{s+1}=0$ with $x_i\in A\,(1\le i\le s+1)$, all such solutions must be trivial, implying that

$$\sum_{x_1\in A}\cdots\sum_{x_{s+1}\in A}\sum_{\chi\in G}\chi(\mathbf{r}\cdot\mathbf{x})\le|G|\sum_{1\le i<j\le s+1}|\{\mathbf{x}\in A^{s+1}:x_i=x_j,\mathbf{r}\cdot\mathbf{x}=0\}|. \tag{2}$$

For any of the $\binom{s+1}{2}$ choices of $1\le i<j\le s+1$, there exists an element $k\in\{1,\ldots,s\}\setminus\{i,j\}$. There are $|A|^{s-1}$ choices of $(x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_{s+1})\in A^s$ where $x_i=x_j$, and given any such choice of $(x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_{s+1})$, $\mathbf{x}\in G^{s+1}$ is a solution of $\mathbf{r}\cdot\mathbf{x}=0$ if and only if $x_k=-\sum_{\substack{l=1\\l\ne k}}^{s+1}r_lx_l$. Thus, for any $1\le i<j\le s+1$,

$$|\{\mathbf{x}\in A^{s+1}:x_i=x_j,\mathbf{r}\cdot\mathbf{x}=0\}|\le|A|^{s-1}. \tag{3}$$

Upon combining (1), (2), and (3), the lemma follows. □

Lemma 3. Let G be a finite Abelian group with $\mathrm{rk}(G)\ge n$, and suppose that $A\subseteq G$ contains no non-trivial solution to $x_1+\cdots+x_s-sx_{s+1}=0$ with $x_i\in A\,(1\le i\le s+1)$. Then,

$$\sum_{\chi\in\widehat{G}} f_1(\chi)f_2(\chi)\cdots f_{s+1}(\chi)\ge|A|^{s+1}-|G||A|^2\big(d(n-1)|G|-|A|\big)^{s-2}.$$

Proof. Note that

$$\begin{aligned}\sum_{\chi\in\widehat{G}} f_1(\chi)\cdots f_{s+1}(\chi)&=f_1(\chi_0)\cdots f_{s+1}(\chi_0)+\sum_{\chi\in\widehat{G}\setminus\{\chi_0\}}f_1(\chi)\cdots f_{s+1}(\chi)\\&=|A|^{s+1}+\sum_{\chi\in\widehat{G}\setminus\{\chi_0\}}f_1(\chi)\cdots f_{s+1}(\chi)\\&\ge|A|^{s+1}-\left|\sum_{\chi\in\widehat{G}\setminus\{\chi_0\}}f_1(\chi)\cdots f_{s+1}(\chi)\right|.\end{aligned} \tag{4}$$

By [2, Lemma 3],

$$\sup_{\chi\in\widehat{G}\setminus\{\chi_0\}}\left|\sum_{x\in A}\chi(x)\right|\le d(n-1)|G|-|A|,$$

and by [4, Corollary on p. 63],

$$\sum_{\chi\in\widehat{G}\setminus\{\chi_0\}}\left|\sum_{x\in A}\chi(x)\right|^2\le\sum_{\chi\in\widehat{G}}\left|\sum_{x\in A}\chi(x)\right|^2=\sum_{x,y\in A}\sum_{\chi\in\widehat{G}}\chi(x-y)=\sum_{\substack{x,y\in A\\x=y}}|G|=|G||A|.$$

Therefore,

$$\begin{aligned}\left|\sum_{\chi\in\widehat{G}\setminus\{\chi_0\}} f_1(\chi)\cdots f_{s+1}(\chi)\right| &\le |A| \sum_{\chi\in\widehat{G}\setminus\{\chi_0\}} |f_1(\chi)\cdots f_s(\chi)| \\ &= |A| \sum_{\chi\in\widehat{G}\setminus\{\chi_0\}} \left|\sum_{x\in A}\chi(x)\right|^s \\ &\le |A|\big(d(n-1)|G|-|A|\big)^{s-2} \sum_{\chi\in\widehat{G}\setminus\{\chi_0\}} \left|\sum_{x\in A}\chi(x)\right|^2 \\ &= |G||A|^2\big(d(n-1)|G|-|A|\big)^{s-2}.\end{aligned} \tag{5}$$

The lemma now follows by combining (4) and (5). □

We are now in a position to prove Theorem 1.

Proof. (of Theorem 1) We proceed by induction on n. We have $d(1) \le 1 \le \mathcal{C}/1^{s-2}$. Suppose now that $n \ge 2$ and that $d(n-1) \le \mathcal{C}/(n-1)^{s-2}$. Let G be a finite Abelian group with $\mathrm{rk}(G) \ge k$, and suppose that $A \subseteq G$ contains no non-trivial solution to $x_1 + \cdots + x_s - sx_{s+1} = 0$ with $x_i \in A\,(1 \le i \le s+1)$. By proving that $|A|/|G| \le \mathcal{C}/n^{s-2}$, we establish the inequality $d(k) \le \mathcal{C}/n^{s-2}$.

Combining Lemmas 2 and 3 yields

$$|A|^{s+1} - |G||A|^2\big(d(n-1)|G|-|A|\big)^{s-2} \le |G|\,|A|^{s-1}\binom{s+1}{2}. \tag{6}$$

We split our analysis into two cases.

- **Case 1.** $\frac{|A|^{s+1}}{2} \le \binom{s+1}{2}|G|\,|A|^{s-1}$

  We may re-write the above inequality as $\frac{|A|}{|G|} \le \sqrt{\frac{s^2+s}{|G|}}$. Because $\mathrm{rk}(G) \ge n$, $|G| \ge 2^n$. Hence,

  $$\frac{|A|}{|G|} \le \sqrt{\frac{s^2+s}{2^n}} = \frac{1}{n^{s-2}}\sqrt{\frac{(s^2+s)n^{2s-4}}{2^n}}.$$

  By considering the first and second derivative, one can show that for $x \ge 1$, $\sqrt{\frac{(s^2+s)x^{2s-4}}{2^x}}$ as a function of x obtains a global maximum of $\left(\frac{2s-4}{e\log(2)}\right)^{s-2}\sqrt{s^2+s}$ when $x = (2s-4)/\log(2)$. Thus, $|A|/|G| \le \mathcal{C}/n^{s-2}$.

- **Case 2.** $\frac{|A|^{s+1}}{2} > \binom{s+1}{2}|G|\,|A|^{s-1}$

  By combining the above inequality with (6), we obtain that

  $$\frac{|A|^{s+1}}{2} < |G||A|^2\big(d(n-1)|G|-|A|\big)^{s-2}.$$

  We may re-write this inequality as

$$\frac{|A|}{|G|} + 2^{\frac{-1}{s-2}} \left(\frac{|A|}{|G|}\right)^{\frac{s-1}{s-2}} < d(n-1) \leq \frac{\mathcal{C}}{(n-1)^{s-2}}. \tag{7}$$

Note that for $x \geq 2$, the function $x\left(\left(\frac{x}{x-1}\right)^{s-2} - 1\right)$ of x is decreasing. Hence, $n\left(\left(\frac{n}{n-1}\right)^{s-2} - 1\right) \leq 2^{s-1} - 2$ and

$$2n^{s-2}\left(\left(\frac{n}{n-1}\right)^{s-2} - 1\right)^{s-2} \leq 2(2^{s-1}-2)^{s-2} \leq \mathcal{C}.$$

Therefore,

$$\left(\frac{n}{n-1}\right)^{s-2} - 1 \leq \left(\frac{\mathcal{C}}{2n^{s-2}}\right)^{\frac{1}{s-2}} = 2^{\frac{-1}{s-2}} \cdot \frac{\mathcal{C}^{\frac{1}{s-2}}}{n},$$

which implies that

$$\frac{\mathcal{C}}{(n-1)^{s-2}} - \frac{\mathcal{C}}{n^{s-2}} \leq 2^{\frac{-1}{s-2}} \cdot \frac{\mathcal{C}^{\frac{s-1}{s-2}}}{n^{s-1}} = 2^{\frac{-1}{s-2}} \left(\frac{\mathcal{C}}{n^{s-2}}\right)^{\frac{s-1}{s-2}}. \tag{8}$$

By (7) and (8), we have

$$\frac{|A|}{|G|} + 2^{\frac{-1}{s-2}} \left(\frac{|A|}{|G|}\right)^{\frac{s-1}{s-2}} < \frac{\mathcal{C}}{n^{s-2}} + 2^{\frac{-1}{s-2}} \left(\frac{\mathcal{C}}{n^{s-2}}\right)^{\frac{s-1}{s-2}}.$$

Since $x + 2^{-1/(s-2)}x^{(s-1)/(s-2)}$ is an increasing function of x, it follows that $|A|/|G| < \mathcal{C}/n^{s-2}$.

The theorem now follows by induction. □

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