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BBP-type formulas, in general bases, for arctangents of real numbers

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Abstract: BBP-type formulas are usually discovered experimentally, one at a time and in specific bases, through computer searches using PSLQ or other integer relations finding algorithms. In this paper, however, we give a systematic analytical derivation of numerous new explicit digit extraction BBP-type formulas for the arctangents of real numbers in general bases. Our method has the clear advantage that the proofs of the formulas are contained in the derivations, whereas in the experimental approach, proofs of discovered formulas have to be found separately. The high point of this paper is perhaps the discovery, for the first time, of a BBP-type formula for $\pi\sqrt{5}$.
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1 Introduction

A BBP-type formula has the remarkable property that it allows the i-th digit of a mathematical constant to be computed without having to compute any of the previous $i - 1$ digits and without requiring ultra high-precision [1, 2]. BBP-type formulas were first introduced in a 1996 paper [3], where a formula of this type for π was given.

Apart from digit extraction, another reason the study of BBP-type formulas has continued to attract attention is that BBP-type constants are conjectured to be either rational or normal to base b [4, 5, 6], that is their base-b digits are randomly distributed.

BBP-type formulas are usually discovered experimentally, one at a time and in specific bases, through computer searches. In this paper, however, we derive explicit digit extraction BBP-type formulas in general bases.

Although most of the formulas presented are new, our method also accomodates a rediscovery of known results.

2 Definitions and notation

The first degree polylogarithm function, with which we are concerned in this paper, is defined by

$$\mathrm{Li}_1[z] = \sum_{k=1}^{\infty} \frac{z^k}{k}, \quad |z| \le 1\,.$$

For $q, x \in \mathbb{R}$, we have the identities

$$\arctan\left(\frac{q\sin x}{1-q\cos x}\right) = \mathrm{Im}\ \mathrm{Li}_1\left[q\exp(ix)\right] = \sum_{k\ge1} \frac{q^k \sin kx}{k}$$

and

$$-\frac{1}{2}\log\left(1-2q\cos x+q^2\right) = \mathrm{Re}\ \mathrm{Li}_1\left[q\exp(ix)\right] = \sum_{k\ge1} \frac{q^k \cos kx}{k}\,.$$

The constants considered in this paper are of the form

$$\begin{aligned} K &= \sum_{m=1}^{n} \left\{\alpha_m \delta_{\gcd(m,n),1} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{im\pi}{n}\right)\right]\right\} \\ &= \sum_{m=1}^{n} \left\{\alpha_m \delta_{\gcd(m,n),1} \arctan\left(\frac{1/\sqrt{t}\sin(m\pi/n)}{1-1/\sqrt{t}\cos(m\pi/n)}\right)\right\} \end{aligned} \tag{1}$$

where $\gcd(j,k)$ is the greatest common divisor of the integers j and k, δ_{jk} is Kronecker's delta symbol, $n \in \{1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60\}$, $t \in \mathbb{Z}^+$ and $\alpha_j \in \{1, 0, -1\}$ for each j. Note that n here belongs to the set of positive integers such that $\sin(j\pi/n)$ and $\cos(j\pi/n)$, $j < n \in \mathbb{Z}^+$, are expressible in terms of radicals.

The scheme for obtaining the BBP-type formulas is as follows:

For each n, we seek all α_j combinations for which K has a degree 1 BBP-type formula, that is, such that K can be written

$$K = \sum_{k\ge0} \frac{1}{b^k} \sum_{j=1}^{l} \frac{a_j}{kl+j}\,,$$

where b, l (base, length respectively) and a_j are integers.

This is accomplished by writing

$$\begin{aligned} K &= \sum_{m=1}^{n} \left\{\alpha_m \delta_{\gcd(m,n),1}\,\mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{im\pi}{n}\right)\right]\right\} \\ &= \sum_{m=1}^{n}\sum_{r\ge1} \left\{\frac{\alpha_m \delta_{\gcd(m,n),1}}{\sqrt{t^r}\,r} \sin\left(\frac{rm\pi}{n}\right)\right\} \\ &= \sum_{r\ge1} \left\{\frac{1}{\sqrt{t^r}\,r} \sum_{m=1}^{n} \left[\alpha_m \delta_{\gcd(m,n),1} \sin\left(\frac{rm\pi}{n}\right)\right]\right\}, \end{aligned} \tag{2}$$

and then choosing only combinations of α_j that give BBP-type formulas.

In order to save space, we will often give the BBP-type formulas using the compact Bailey's P-notation [2]:

$$P(s,b,l,A) \equiv \sum_{k\geq 0} \frac{1}{b^k} \sum_{j=1}^{l} \frac{a_j}{(kl+j)^s}\,,$$

where s, b and l are integers, and $A = (a_1, a_2, \ldots, a_l)$ is a vector of integers.

3 n=1: Base t formulas

There are no base t length 1 or length 2 arctangent formulas since $\mathrm{Li}_1[1/\sqrt{t}\exp i\pi]$ is a real number for $t > 0$.

4 n=2: Base t^2 formulas

Choosing $n = 2$ in Eq. (2), we find

$$\begin{aligned} K &= \alpha_1 \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{i\pi}{2}\right)\right] \\ &= \sum_{r\geq 1}\left\{\frac{1}{\sqrt{t^r}\,r}\left[\alpha_1 \sin\left(\frac{r\pi}{2}\right)\right]\right\}, \end{aligned} \tag{3}$$

from which follows immediately the BBP-type formula

$$\begin{aligned} \sqrt{t^3}\arctan\left(\frac{1}{\sqrt{t}}\right) &= \sum_{k\geq 0}\frac{1}{(t^2)^k}\left[\frac{t}{(4k+1)} - \frac{1}{(4k+3)}\right] \\ &= P(1, t^2, 4, (t, 0, -1, 0)) \end{aligned} \tag{4}$$

5 $n = 3$: Bases t^3 and t^6 formulas

Choosing $n = 3$ in Eq. (2), we find

$$\begin{aligned} K &= \alpha_1 \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{i\pi}{3}\right)\right] + \alpha_2 \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{2i\pi}{3}\right)\right] \\ &= \sum_{r\geq 1}\left\{\frac{1}{\sqrt{t^r}\,r}\left[\alpha_1 \sin\left(\frac{r\pi}{3}\right) + \alpha_2 \sin\left(\frac{2r\pi}{3}\right)\right]\right\} \end{aligned} \tag{5}$$

All combinations of $\alpha_1, \alpha_2 \in \{1, 0, -1\}$ give BBP-type formulas for K. We therefore consider the combinations in turn:

$$[\alpha_1, \alpha_2] = \begin{cases} [1,0] \\ [0,1] \\ [1,1] \\ [1,-1] \end{cases} \tag{6}$$

5.1 Case $[\alpha_1, \alpha_2] = [1, 0]$

With $\alpha_1 = 1 = 1 - \alpha_2$ in Eq. (5), we have

$$\begin{aligned} K &= \operatorname{Im} \operatorname{Li}_1 \left[\frac{1}{\sqrt{t}} \exp\left(\frac{i\pi}{3}\right) \right] \\ &= \arctan\left(\frac{\sqrt{3}}{2\sqrt{t}-1}\right) \\ &= \sum_{r\geq 1} \frac{1}{\sqrt{t^r} r} \sin\left(\frac{r\pi}{3}\right) \\ &= \frac{\sqrt{3}}{2t^3} \sum_{k\geq 0} \frac{1}{(t^3)^k} \left[\frac{t^{-1/2}t^3}{6k+1} + \frac{t^2}{6k+2} - \frac{t}{6k+4} - \frac{\sqrt{t}}{6k+5} \right] \end{aligned} \tag{7}$$

Note that Eq. (7) is BBP-type only if $\sqrt{t} \in \mathbb{Q}$. Thus replacing t with t^2, we obtain the following BBP-type formula:

$$\begin{aligned} & 2\sqrt{3} \arctan\left(\frac{\sqrt{3}}{2t-1}\right) \\ &= \frac{3}{t^5} \sum_{k\geq 0} \frac{1}{(t^6)^k} \left[\frac{t^4}{6k+1} + \frac{t^3}{6k+2} - \frac{t}{6k+4} - \frac{1}{6k+5} \right] \\ &= \frac{3}{t^5} P(1, t^6, 6, (t^4, t^3, 0, -t, -1, 0)) \,. \end{aligned} \tag{8}$$

Henceforth we shall make such $t \to t^2$ replacements without further comments, as the need arises.

5.2 Case $[\alpha_1, \alpha_2] = [0, 1]$

With $\alpha_1 = 0 = \alpha_2 - 1$ in Eq. (5), we have

$$\begin{aligned} & 2\sqrt{3} \arctan\left(\frac{\sqrt{3}}{2t+1}\right) \\ &= \frac{3}{t^5} \sum_{k\geq 0} \frac{1}{(t^6)^k} \left[\frac{t^4}{6k+1} - \frac{t^3}{6k+2} + \frac{t}{6k+4} - \frac{1}{6k+5} \right] \\ &= \frac{3}{t^5} P(1, t^6, 6, (t^4, -t^3, 0, t, -1, 0)) \,. \end{aligned} \tag{9}$$

5.3 Case $[\alpha_1, \alpha_2] = [1, 1]$

With $\alpha_1 = 1 = \alpha_2$ in Eq. (5), we have

$$
\begin{aligned}
&\sqrt{3}\arctan\left(\frac{\sqrt{3t}}{t-1}\right)\\
&=\frac{3}{t^3}\sum_{k\geq 0}\frac{1}{(t^3)^k}\left[\frac{\sqrt{t^5}}{6k+1}-\frac{\sqrt{t}}{6k+5}\right]\\
&=\frac{3}{\sqrt{t^5}}P(1,t^3,6,(t^2,0,0,0,-1,0))
\end{aligned}
\tag{10}
$$

5.4 Case $[\alpha_1, \alpha_2] = [1, -1]$

With $\alpha_1 = 1 = -\alpha_2$ in Eq. (5), we have

$$
\begin{aligned}
&2\sqrt{3}\arctan\left(\frac{\sqrt{3}}{2t+1}\right)\\
&=\frac{3}{t^2}\sum_{k\geq 0}\frac{1}{(t^3)^k}\left[\frac{t}{3k+1}-\frac{1}{3k+2}\right]\\
&=\frac{3}{t^2}P(1,t^3,3,(t,-1,0))
\end{aligned}
\tag{11}
$$

Note that Eq. (11) is the base t^3, length 3 version of Eq. (9).

6 n=4: Bases t^4 and 2^4t^8 formulas

Choosing $n = 4$ in Eq. (2), we find

$$
\begin{aligned}
K &= \alpha_1 \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{i\pi}{4}\right)\right] + \alpha_3 \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{3i\pi}{4}\right)\right]\\
&= \sum_{r\geq 1}\left\{\frac{1}{\sqrt{t^r}r}\left[\alpha_1\sin\left(\frac{r\pi}{4}\right)+\alpha_3\sin\left(\frac{3r\pi}{4}\right)\right]\right\}
\end{aligned}
\tag{12}
$$

As in the previous case of $n = 3$, all combinations of $\alpha_1, \alpha_3 \in \{1, 0, -1\}$ give BBP-type formulas for K. We will consider, in turn, the combinations $[\alpha_1, \alpha_3] = [1, 0]$, $[\alpha_1, \alpha_3] = [0, 1]$, $[\alpha_1, \alpha_3] = [1, 1]$ and $[\alpha_1, \alpha_3] = [1, -1]$.

6.1 Case $[\alpha_1, \alpha_3] = [1, 0]$

With $\alpha_1 = 1 = 1 - \alpha_3$ in Eq. (12), we have

$$
\begin{aligned}
&\arctan\left(\frac{1}{2t-1}\right)\\
&=\frac{1}{16t^7}P(1,16t^8,8,(8t^6,8t^5,4t^4,0,-2t^2,-2t,-1,0))\,.
\end{aligned}
\tag{13}
$$

6.2 Case $[\alpha_1, \alpha_3] = [0, 1]$

With $\alpha_1 = 0 = \alpha_3 - 1$ in Eq. (12), we have

$$\begin{aligned} &\arctan\left(\frac{1}{2t+1}\right) \\ &= \frac{1}{16t^7} P(1, 16t^8, 8, (8t^6, -8t^5, 4t^4, 0, -2t^2, 2t, -1, 0)) \,. \end{aligned} \tag{14}$$

6.3 Case $[\alpha_1, \alpha_3] = [1, 1]$

With $\alpha_1 = 1 = \alpha_3$ in Eq. (12), we have

$$\begin{aligned} &\sqrt{2}\arctan\left(\frac{\sqrt{2t}}{t-1}\right) \\ &= \frac{2}{\sqrt{t^7}} P(1, t^4, 8, (t^3, 0, t^2, 0, -t, 0, -1, 0)) \,. \end{aligned} \tag{15}$$

6.4 Case $[\alpha_1, \alpha_3] = [1, -1]$

Here Eq. (4) is reproduced.

7 n=5: Base t^5 formulas

Choosing $n = 5$ in Eq. (2), we find

$$\begin{aligned} K &= \alpha_1 \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{i\pi}{5}\right)\right] + \alpha_2 \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{2i\pi}{5}\right)\right] \\ &\quad + \alpha_3 \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{3i\pi}{5}\right)\right] + \alpha_4 \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{4i\pi}{5}\right)\right] \\ &= \sum_{r\geq 1}\left\{\frac{1}{\sqrt{t^r}r}\left[\alpha_1\sin\left(\frac{r\pi}{5}\right) + \alpha_2\sin\left(\frac{2r\pi}{5}\right) + \alpha_3\sin\left(\frac{3r\pi}{5}\right) + \alpha_4\sin\left(\frac{4r\pi}{5}\right)\right]\right\} \end{aligned} \tag{16}$$

Out of all the 40 independent combinations of $\alpha_j, j = 1, 2, 3, 4$, none gives a BBP-type series for K.

8 n=6: Base t^6 formulas

Choosing $n = 6$ in Eq. (2), we find

$$\begin{aligned} K &= \alpha_1 \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{i\pi}{6}\right)\right] + \alpha_5 \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{5i\pi}{6}\right)\right] \\ &= \sum_{r\geq 1}\left\{\frac{1}{\sqrt{t^r}r}\left[\alpha_1\sin\left(\frac{r\pi}{6}\right) + \alpha_5\sin\left(\frac{5r\pi}{6}\right)\right]\right\} \end{aligned} \tag{17}$$

Out of the four independent combinations of $\alpha_1, \alpha_5 \in \{1, 0, -1\}$, two, namely $[\alpha_1, \alpha_5] = [1, 1]$ and $[\alpha_1, \alpha_5] = [1, -1]$, give BBP-type formulas for K.

8.1 Case $[\alpha_1, \alpha_5] = [1, 1]$

With $\alpha_1 = 1 = \alpha_5$ in Eq. (17), we have

$$\begin{aligned} &\arctan\left(\frac{\sqrt{t}}{t-1}\right) \\ &= \frac{1}{\sqrt{t^{11}}} P(1, t^6, 12, (t^5, 0, 2t^4, 0, t^3, 0, -t^2, 0, -2t, 0, -1, 0)) \,. \end{aligned} \tag{18}$$

8.2 Case $[\alpha_1, \alpha_5] = [1, -1]$

With $\alpha_1 = 1 = -\alpha_5$ in Eq. (17), Eq. (8) is merely reproduced.

9 $n = 8$: Base t^8 formulas

Choosing $n = 8$ in Eq. (2), we find

$$\begin{aligned} K &= \alpha_1 \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{i\pi}{8}\right)\right] + \alpha_3 \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{3i\pi}{8}\right)\right] \\ &\quad + \alpha_5 \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{5i\pi}{8}\right)\right] + \alpha_7 \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{7i\pi}{8}\right)\right] \\ &= \sum_{r\geq 1}\left\{\frac{1}{\sqrt{t^r}\,r}\left[\alpha_1\sin\left(\frac{r\pi}{8}\right) + \alpha_3\sin\left(\frac{3r\pi}{8}\right) + \alpha_5\sin\left(\frac{5r\pi}{8}\right) + \alpha_7\sin\left(\frac{7r\pi}{8}\right)\right]\right\} \end{aligned} \tag{19}$$

Out of all the 40 independent combinations of α_1, α_3, α_5 and α_7, two give BBP-type series for K. The BBP-type formulas for K are found under $[\alpha_1, \alpha_3, \alpha_5, \alpha_7] = [1, 1, -1, -1]$ and $[\alpha_1, \alpha_3, \alpha_5, \alpha_7] = [1, -1, 1, -1]$.

9.1 Case $[\alpha_1, \alpha_3, \alpha_5, \alpha_7] = [1, 1, -1, -1]$

Replicates Eq. (15)

9.2 Case $[\alpha_1, \alpha_3, \alpha_5, \alpha_7] = [1, -1, 1, -1]$

Replicates Eq. (4)

10 $n = 10$: Base t^{10} formulas

Choosing $n = 10$ in Eq. (2), we find

$$\begin{aligned} K &= \alpha_1 \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{i\pi}{10}\right)\right] + \alpha_3 \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{3i\pi}{10}\right)\right] \\ &\quad + \alpha_7 \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{7i\pi}{10}\right)\right] + \alpha_9 \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{9i\pi}{10}\right)\right] \\ &= \sum_{r\geq 1}\left\{\frac{1}{\sqrt{t^r}\,r}\left[\alpha_1\sin\left(\frac{r\pi}{10}\right) + \alpha_3\sin\left(\frac{3r\pi}{10}\right) + \alpha_7\sin\left(\frac{7r\pi}{10}\right) + \alpha_9\sin\left(\frac{9r\pi}{10}\right)\right]\right\} \end{aligned} \tag{20}$$

Out of all the 40 independent combinations of α_1, α_3, α_7 and α_9, two give BBP-type series for K. The BBP-type formulas for K are found under $[\alpha_1,\alpha_3,\alpha_7,\alpha_9]=[1,1,1,1]$ and $[\alpha_1,\alpha_3,\alpha_7,\alpha_9]=[1,-1,-1,1]$.

10.1 Case $[\alpha_1,\alpha_3,\alpha_7,\alpha_9]=[1,1,1,1]$

Using $\alpha_1=1=\alpha_3=\alpha_7=\alpha_9$ in Eq. (20) we find

$$\begin{aligned}&\sqrt{5}\arctan\left\{\sqrt{5t}\left(\frac{t-1}{t^2-3t+1}\right)\right\}+(\delta_{t1}+\delta_{t2})\pi\sqrt{5}\\&=\frac{5}{\sqrt{t^{19}}}P(1,t^{10},20,(t^9,0,t^8,0,0,0,t^6,0,t^5,0,\\&\qquad -t^4,0,-t^3,0,0,0,-t,0,-1,0))\end{aligned} \tag{21}$$

10.2 Case $[\alpha_1,\alpha_3,\alpha_7,\alpha_9]=[1,-1,-1,1]$

Using $\alpha_1=1=-\alpha_3=-\alpha_7=\alpha_9$ in Eq. (20) we find

$$\begin{aligned}&\arctan\left\{\sqrt{t}\left(\frac{t-1}{t^2-t+1}\right)\right\}\\&=\frac{1}{\sqrt{t^{19}}}P(1,t^{10},20,(t^9,0,-t^8,0,-4t^7,0,-t^6,0,\\&\qquad t^5,0,-t^4,0,t^3,0,4t^2,0,t,0,-1,0))\end{aligned} \tag{22}$$

11 $n=12$: Bases t^{12} and $2^{12}t^{24}$ formulas

Choosing $n=12$ in Eq. (2), we find

$$\begin{aligned}K&=\alpha_1\mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{i\pi}{12}\right)\right]+\alpha_5\mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{5i\pi}{12}\right)\right]\\&\quad+\alpha_7\mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{7i\pi}{12}\right)\right]+\alpha_{11}\mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{11i\pi}{12}\right)\right]\\&=\sum_{r\ge1}\left\{\frac{1}{\sqrt{t^r}r}\left[\alpha_1\sin\left(\frac{r\pi}{12}\right)+\alpha_5\sin\left(\frac{5r\pi}{12}\right)+\alpha_7\sin\left(\frac{7r\pi}{12}\right)+\alpha_{11}\sin\left(\frac{11r\pi}{12}\right)\right]\right\}\end{aligned} \tag{23}$$

Out of all the 40 independent combinations of $\alpha_j, j=1,5,7,11$, BBP-type series exist for the following α_j combinations:

$$[\alpha_1,\alpha_5,\alpha_7,\alpha_{11}]=\begin{cases}[1,1,1,1]\\[1,1,-1,-1]\\[1,-1,1,-1]\\[1,-1,-1,1]\\[1,0,1,0]\\[1,0,-1,0]\\[0,1,0,1]\\[0,1,0,-1]\end{cases} \tag{24}$$

11.1 Case $[\alpha_1, \alpha_5, \alpha_7, \alpha_{11}] = [1, 1, 1, 1]$

$$\begin{aligned} &\sqrt{6}\arctan\left\{\sqrt{6t}\left(\frac{t-1}{t^2-3t+1}\right)\right\} + \pi\sqrt{6}\left(\delta_{t1}+\delta_{t2}\right) \\ &= \frac{6}{\sqrt{t^{23}}}P(1, t^{12}, 24, (t^{11}, 0, 0, 0, t^9, 0, t^8, \\ &\quad 0, 0, 0, t^6, 0, -t^5, 0, 0, 0, -t^3, 0, -t^2, 0, 0, 0, -1, 0)) \end{aligned} \tag{25}$$

11.2 Case $[\alpha_1, \alpha_5, \alpha_7, \alpha_{11}] = [1, 1, -1, -1]$

Replicates Eq. (18)

11.3 Case $[\alpha_1, \alpha_5, \alpha_7, \alpha_{11}] = [1, -1, 1, -1]$

Replicates Eq. (8).

11.4 Case $[\alpha_1, \alpha_5, \alpha_7, \alpha_{11}] = [1, -1, -1, 1]$

$$\begin{aligned} &\sqrt{2}\arctan\left\{\sqrt{2t}\left(\frac{t-1}{t^2-t+1}\right)\right\} \\ &= \frac{2}{\sqrt{t^{23}}}P(1, t^{12}, 24, (t^{11}, 0, -2t^{10}, 0, -t^9, 0, -t^8, 0, \\ &\quad -2t^7, 0, t^6, 0, -t^5, 0, 2t^4, 0, t^3, 0, t^2, 0, 2t, 0, -1, 0)) \end{aligned} \tag{26}$$

11.5 Case $[\alpha_1, \alpha_5, \alpha_7, \alpha_{11}] = [1, 0, 1, 0]$

$$\begin{aligned} &\sqrt{3}\arctan\left\{\sqrt{3}\left(\frac{2t-1}{4t^2-2t-1}\right)\right\} \\ &= \frac{3}{4096t^{23}}P(1, 4096t^{24}, 24, (2048t^{22}, 0, 0, 1024t^{19}, 512t^{18}, 0, \\ &256t^{16}, 256t^{15}, 0, 0, 64t^{12}, 0, -32t^{10}, 0, 0, -16t^7, \\ &-8t^6, 0, -4t^4, -4t^3, 0, 0, -1, 0)) \end{aligned} \tag{27}$$

11.6 Case $[\alpha_1, \alpha_5, \alpha_7, \alpha_{11}] = [1, 0, -1, 0]$

$$\begin{aligned} &\arctan\left\{\frac{1}{t}\left(\frac{t-1}{2t-1}\right)\right\} \\ &= \frac{1}{4096t^{23}}P(1, 4096t^{24}, 24, (2048t^{22}, -2048t^{21}, \\ &\quad -2048t^{20}, 0, -512t^{18}, -1024t^{17}, -256t^{16}, 0, \\ &\quad -256t^{14}, -128t^{13}, 64t^{12}, 0, -32t^{10}, \\ &\quad 32t^9, 32t^8, 0, 8t^6, 16t^5, \\ &\quad 4t^4, 0, 4t^2, 2t, -1, 0)) \end{aligned} \tag{28}$$

11.7 Case $[\alpha_1, \alpha_5, \alpha_7, \alpha_{11}] = [0, 1, 0, 1]$

$$\begin{aligned}
&\sqrt{3}\arctan\left\{\sqrt{3}\left(\frac{2t+1}{4t^2+2t-1}\right)\right\} \\
&= \frac{3}{4096t^{23}} P(1, 4096t^{24}, 24, (2048t^{22}, 0, 0, -1024t^{19}, \\
&\quad 512t^{18}, 0, 256t^{16}, -256t^{15}, 0, 0, 64t^{12}, 0, -32t^{10}, \\
&\quad 0, 0, 16t^7, -8t^6, 0, -4t^4, 4t^3, 0, 0, -1, 0))
\end{aligned} \tag{29}$$

11.8 Case $[\alpha_1, \alpha_5, \alpha_7, \alpha_{11}] = [0, 1, 0, -1]$

$$\begin{aligned}
&\arctan\left\{\frac{1}{t}\left(\frac{t+1}{2t+1}\right)\right\} \\
&= \frac{1}{4096t^{23}} P(1, 4096t^{24}, 24, (2048t^{22}, 2048t^{21}, -2048t^{20}, \\
&0, -512t^{18}, 1024t^{17}, -256t^{16}, 0, -256t^{14}, 128t^{13}, \\
&64t^{12}, 0, -32t^{10}, -32t^9, 32t^8, 0, 8t^6, \\
&-16t^5, 4t^4, 0, 4t^2, -2t, -1, 0))
\end{aligned} \tag{30}$$

We remark here that the binary BBP formulas that result by evaluating at $t = 2^{p-1}$, $p \in \mathbb{Z}^+$ Eqs. (27), (28), (29) and (30), have been discussed elsewhere (see [7]).

12 $n = 15$: Bases t^{15} and t^{30} formulas

Choosing $n = 15$ in Eq. (2), we find

$$\begin{aligned}
K &= \alpha_1 \mathrm{Im\,Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{i\pi}{15}\right)\right] + \alpha_2 \mathrm{Im\,Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{2i\pi}{15}\right)\right] \\
&+ \alpha_4 \mathrm{Im\,Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{4i\pi}{15}\right)\right] + \alpha_7 \mathrm{Im\,Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{7i\pi}{15}\right)\right] \\
&+ \alpha_8 \mathrm{Im\,Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{8i\pi}{15}\right)\right] + \alpha_{11} \mathrm{Im\,Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{11i\pi}{15}\right)\right] \\
&+ \alpha_{13} \mathrm{Im\,Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{13i\pi}{15}\right)\right] + \alpha_{14} \mathrm{Im\,Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{14i\pi}{15}\right)\right] \\
&= \sum_{r\geq 1}\left\{\frac{1}{\sqrt{t^r}r}\left[\alpha_1 \sin\left(\frac{r\pi}{15}\right) + \alpha_2 \sin\left(\frac{2r\pi}{15}\right) + \alpha_4 \sin\left(\frac{4r\pi}{15}\right)\right.\right. \\
&\quad + \alpha_7 \sin\left(\frac{7r\pi}{15}\right) + \alpha_8 \sin\left(\frac{8r\pi}{15}\right) + \alpha_{11} \sin\left(\frac{11r\pi}{15}\right) \\
&\quad \left.\left. + \alpha_{13} \sin\left(\frac{13r\pi}{15}\right) + \alpha_{14} \sin\left(\frac{14r\pi}{15}\right)\right]\right\}
\end{aligned} \tag{31}$$

BBP-type series exist for the following α_j combinations:

$$[\alpha_1,\alpha_2,\alpha_4,\alpha_7,\alpha_8,\alpha_{11},\alpha_{13},\alpha_{14}] = \begin{cases} [1,-1,-1,-1,-1,-1,-1,1] \\ [1,1,1,-1,1,-1,-1,-1] \\ [1,1,-1,1,1,-1,1,1] \\ [1,-1,1,1,-1,-1,1,-1] \\ [1,0,0,1,0,-1,1,0] \\ [1,0,0,-1,0,-1,-1,0] \\ [0,1,1,0,1,0,0,-1] \\ [0,1,-1,0,1,0,0,1] \end{cases} \tag{32}$$

12.1 Case $[\alpha_1,\alpha_2,\alpha_4,\alpha_7,\alpha_8,\alpha_{11},\alpha_{13},\alpha_{14}] = [1,-1,-1,-1,-1,-1,-1,1]$

$$\begin{aligned}&\sqrt{15}\arctan\left\{\sqrt{15t}\left(\frac{t^3-3t^2+3t-1}{t^4-8t^3+13t^2-8t+1}\right)\right\}+\pi\sqrt{15}\sum_{j=1}^{6}\delta_{tj}\\ &=\frac{15}{\sqrt{t^{29}}}P(1,t^{15},30,(t^{14},0,0,0,0,0,-t^{11},0,0,0,\\ &\quad -t^9,0,-t^8,0,0,0,t^6,0,t^5,0,0,0,t^3,0,0,0,0,0,-1,0))\end{aligned} \tag{33}$$

12.2 Case $[\alpha_1,\alpha_2,\alpha_4,\alpha_7,\alpha_8,\alpha_{11},\alpha_{13},\alpha_{14}] = [1,1,1,-1,1,-1,-1,-1]$

$$\begin{aligned}&2\sqrt{15}\arctan\left\{t\sqrt{15}\left(\frac{t^2-1}{2t^4-4t^2-t^3-t+2}\right)\right\}+2\pi\sqrt{15}\,\delta_{t1}\\ &=\frac{15}{t^{14}}P(1,t^{15},15,(t^{13},t^{12},0,t^{10},0,0,-t^7,t^6,0,0,-t^3,0,-t,-1,0))\end{aligned} \tag{34}$$

12.3 Case $[\alpha_1,\alpha_2,\alpha_4,\alpha_7,\alpha_8,\alpha_{11},\alpha_{13},\alpha_{14}] = [1,1,-1,1,1,-1,1,1]$

$$\begin{aligned}&\sqrt{3}\arctan\left\{\sqrt{3t}\left(\frac{t^3-t^2+t-1}{t^4-2t^3+t^2-2t+1}\right)\right\}+\pi\sqrt{3}\,\delta_{t1}\\ &=\frac{3}{\sqrt{t^{29}}}P(1,t^{15},30,(t^{14},0,0,0,4t^{12},0,t^{11},0,0,0,\\ &\quad -t^9,0,t^8,0,0,0,-t^6,0,t^5,0,0,0,-t^3,0,-4t^2,0,0,0,-1,0))\end{aligned} \tag{35}$$

12.4 Case $[\alpha_1,\alpha_2,\alpha_4,\alpha_7,\alpha_8,\alpha_{11},\alpha_{13},\alpha_{14}] = [1,-1,1,1,-1,-1,1,-1]$

$$\begin{aligned}&2\sqrt{3}\arctan\left\{\sqrt{3}\left(\frac{t^3-t^2+1}{2t^4-t^3-t^2+2t-1}\right)\right\}\\ &=\frac{3}{t^{14}}P(1,t^{15},15,(t^{13},-t^{12},0,t^{10},4t^9,0,t^7,\\ &\quad -t^6,0,-4t^4,-t^3,0,t,-1,0))\end{aligned} \tag{36}$$

12.5 Case $[\alpha_1, \alpha_2, \alpha_4, \alpha_7, \alpha_8, \alpha_{11}, \alpha_{13}, \alpha_{14}] = [1, 0, 0, 1, 0, -1, 1, 0]$

$$\begin{aligned} & 2\sqrt{3}\arctan\left\{\sqrt{3}\left(\frac{t^3+t^2-1}{2t^4+t^3-t^2-2t-1}\right)\right\} + 2\pi\sqrt{3}\,\delta_{t1} \\ &= \frac{3}{t^{29}}P(1, t^{30}, 30, (t^{28}, t^{27}, 0, -t^{25}, 4t^{24}, 0, \\ & t^{22}, t^{21}, 0, 4t^{19}, -t^{18}, 0, t^{16}, t^{15}, 0, -t^{13}, -t^{12}, \\ & 0, t^{10}, -4t^9, 0, -t^7, -t^6, 0, -4t^4, t^3, 0, -t, -1, 0)) \end{aligned} \tag{37}$$

12.6 Case $[\alpha_1, \alpha_2, \alpha_4, \alpha_7, \alpha_8, \alpha_{11}, \alpha_{13}, \alpha_{14}] = [1, 0, 0, -1, 0, -1, -1, 0]$

$$\begin{aligned} & 2\sqrt{15}\arctan\left\{t\sqrt{15}\left(\frac{t^2-1}{2t^4+t^3-4t^2+t+2}\right)\right\} \\ &= \frac{15}{t^{29}}P(1, t^{30}, 30, (t^{28}, -t^{27}, 0, -t^{25}, 0, 0, \\ & - t^{22}, -t^{21}, 0, 0, -t^{18}, 0, -t^{16}, t^{15}, 0, -t^{13}, \\ & t^{12}, 0, t^{10}, 0, 0, t^7, t^6, 0, 0, t^3, 0, t, -1, 0)) \end{aligned} \tag{38}$$

12.7 Case $[\alpha_1, \alpha_2, \alpha_4, \alpha_7, \alpha_8, \alpha_{11}, \alpha_{13}, \alpha_{14}] = [0, 1, 1, 0, 1, 0, 0, -1]$

$$\begin{aligned} & 2\sqrt{15}\arctan\left\{t\sqrt{15}\left(\frac{t^2-1}{2t^4-4t^2-t^3-t+2}\right)\right\} + 2\pi\sqrt{15}\,\delta_{t1} \\ &= \frac{15}{t^{29}}P(1, t^{30}, 30, (t^{28}, t^{27}, 0, t^{25}, 0, 0, \\ & - t^{22}, t^{21}, 0, 0, -t^{18}, 0, -t^{16}, -t^{15}, 0, t^{13}, \\ & t^{12}, 0, t^{10}, 0, 0, -t^7, t^6, 0, 0, -t^3, 0, -t, -1, 0)) \end{aligned} \tag{39}$$

12.8 Case $[\alpha_1, \alpha_2, \alpha_4, \alpha_7, \alpha_8, \alpha_{11}, \alpha_{13}, \alpha_{14}] = [0, 1, -1, 0, 1, 0, 0, 1]$

$$\begin{aligned} & 2\sqrt{3}\arctan\left\{\sqrt{3}\left(\frac{t^3-t^2+1}{2t^4-t^3-t^2+2t-1}\right)\right\} \\ &= \frac{3}{t^{29}}P(1, t^{30}, 30, (t^{28}, -t^{27}, 0, t^{25}, 4t^{24}, 0, \\ & t^{22}, -t^{21}, 0, -4t^{19}, -t^{18}, 0, t^{16}, -t^{15}, 0, t^{13}, \\ & - t^{12}, 0, t^{10}, 4t^9, 0, t^7, -t^6, 0, -4t^4, -t^3, 0, t, -1, 0)) \end{aligned} \tag{40}$$

Note that Eq. (40) is an expanded version of Eq. (36).

13 $n = 20$: Bases t^{20} and $2^{20}t^{40}$ formulas

Choosing $n = 20$ in Eq. (2), we find

$$
\begin{aligned}
K &= \alpha_1 \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{i\pi}{20}\right)\right] + \alpha_3 \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{3i\pi}{20}\right)\right] \\
&\quad + \alpha_7 \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{7i\pi}{20}\right)\right] + \alpha_9 \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{9i\pi}{20}\right)\right] \\
&\quad + \alpha_{11} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{11i\pi}{20}\right)\right] + \alpha_{13} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{13i\pi}{20}\right)\right] \\
&\quad + \alpha_{17} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{17i\pi}{20}\right)\right] + \alpha_{19} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{19i\pi}{20}\right)\right] \\
&= \sum_{r\geq 1}\left\{\frac{1}{\sqrt{t^r}r}\left[\alpha_1\sin\left(\frac{r\pi}{20}\right) + \alpha_3\sin\left(\frac{3r\pi}{20}\right) + \alpha_7\sin\left(\frac{7r\pi}{20}\right)\right.\right. \\
&\qquad + \alpha_9\sin\left(\frac{9r\pi}{20}\right) + \alpha_{11}\sin\left(\frac{11r\pi}{20}\right) + \alpha_{13}\sin\left(\frac{13r\pi}{20}\right) \\
&\qquad \left.\left. + \alpha_{17}\sin\left(\frac{17r\pi}{20}\right) + \alpha_{19}\sin\left(\frac{19r\pi}{20}\right)\right]\right\}
\end{aligned} \tag{41}
$$

BBP-type series exist for the following α_j combinations:

$$
[\alpha_1,\alpha_3,\alpha_7,\alpha_9,\alpha_{11},\alpha_{13},\alpha_{17},\alpha_{19}] = \begin{cases}
[1,1,-1,1,1,-1,1,1] \\
[1,-1,1,1,1,1,-1,1] \\
[1,1,1,1,-1,-1,-1,-1] \\
[1,-1,-1,1,-1,1,1,-1] \\
[1,0,1,1,0,0,-1,0] \\
[1,0,-1,1,0,0,1,0] \\
[0,1,0,0,1,-1,0,1] \\
[0,1,0,0,-1,-1,0,-1]
\end{cases} \tag{42}
$$

13.1 Case $[\alpha_1,\alpha_3,\alpha_7,\alpha_9,\alpha_{11},\alpha_{13},\alpha_{17},\alpha_{19}] = [1,1,-1,1,1,-1,1,1]$

$$
\begin{aligned}
&\sqrt{2}\arctan\left\{\sqrt{2t}\left(\frac{t^3-1}{t^4-t^3-t^2-t+1}\right)\right\} + \pi\sqrt{2}\,\delta_{t1} \\
&= \frac{2}{\sqrt{t^{39}}}P(1,t^{20},40,(t^{19},0,t^{18},0,4t^{17},0,-t^{16},0,t^{15},0,t^{14},0,-t^{13},0,4t^{12},0,t^{11}, \\
&\qquad 0,t^{10},0,-t^9,0,-t^8,0,-4t^7,0,t^6,0,-t^5,0,-t^4,0,t^3,0,-4t^2,0,-t,0,-1,0))
\end{aligned} \tag{43}
$$

13.2 Case $[\alpha_1,\alpha_3,\alpha_7,\alpha_9,\alpha_{11},\alpha_{13},\alpha_{17},\alpha_{19}] = [1,-1,1,1,1,1,-1,1]$

$$
\begin{aligned}
&\sqrt{5}\arctan\left\{\sqrt{10t}\left(\frac{t^3-2t^2+2t-1}{t^4-5t^3+7t^2-5t+1}\right)\right\} + (\delta_{t1}+\delta_{t2}+\delta_{t3})\pi\sqrt{5} \\
&= \frac{5\sqrt{2}}{\sqrt{t^{39}}}P(1,t^{20},40,(t^{19},0,-t^{18},0,0,0,t^{16},0,t^{15},0,t^{14},0,t^{13},0,0,0,-t^{11},0,t^{10},0, \\
&\qquad -t^9,0,t^8,0,0,0,-t^6,0,-t^5,0,-t^4,0,-t^3,0,0,0,t,0,-1,0))
\end{aligned} \tag{44}
$$

13.3 Case $[\alpha_1, \alpha_3, \alpha_7, \alpha_9, \alpha_{11}, \alpha_{13}, \alpha_{17}, \alpha_{19}] = [1, 1, 1, 1, -1, -1, -1, -1]$

$$\begin{aligned}&\sqrt{5}\arctan\left\{t\sqrt{5}\left(\frac{t^2-1}{t^4-3t^2+1}\right)\right\} + \pi\sqrt{5}\,\delta_{t1}\\&= \frac{5}{t^{19}}P(1, t^{20}, 20, (t^{18}, 0, t^{16}, 0, 0, 0, t^{12}, 0, t^{10}, 0, -t^8, 0, -t^6, 0, 0, 0, -t^2, 0, -1, 0))\end{aligned} \tag{45}$$

Note that Eq. (45) and Eq. (21) are identical.

Addition of Eqs. (44) and (45), both evaluated at $t = 2$, gives a very interesting binary BBP-type formula for $\pi\sqrt{5}$:

$$\begin{aligned}\pi\sqrt{5} = \frac{5}{2^{19}}P(1, 2^{20}, 40, (&2^{19}, 2^{19}, -2^{18}, 0, 0, 2^{17}, 2^{16}, 0,\\&2^{15}, 0, 2^{14}, 0, 2^{13}, 2^{13}, 0, 0, -2^{11}, 2^{11}, 2^{10}, 0, -2^9, -2^9, 2^8, 0, 0,\\&-2^7, -2^6, 0, -2^5, 0, -2^4, 0, -2^3, -2^3, 0, 0, 2, -2, -1, 0))\end{aligned} \tag{46}$$

13.4 Case $[\alpha_1, \alpha_3, \alpha_7, \alpha_9, \alpha_{11}, \alpha_{13}, \alpha_{17}, \alpha_{19}] = [1, -1, -1, 1, -1, 1, 1, -1]$

Replicates Eq. (22)

13.5 Case $[\alpha_1, \alpha_3, \alpha_7, \alpha_9, \alpha_{11}, \alpha_{13}, \alpha_{17}, \alpha_{19}] = [1, 0, 1, 1, 0, 0, -1, 0]$

$$\begin{aligned}&\sqrt{5}\arctan\left\{t\sqrt{5}\left(\frac{2t^2-1}{4t^4-2t^3-2t^2-t+1}\right)\right\}\\&= \frac{5}{1048576t^{39}}P(1, 1048576t^{40}, 40, (524288t^{38}, 524288t^{37},\\&-262144t^{36}, 0, 0, 131072t^{33}, 65536t^{32}, 0, 32768t^{30}, 0,\\&16384t^{28}, 0, 8192t^{26}, 8192t^{25}, 0, 0, -2048t^{22},\\&2048t^{21}, 1024t^{20}, 0, -512t^{18}, -512t^{17},\\&256t^{16}, 0, 0, -128t^{13}, -64t^{12}, 0, -32t^{10},\\&0, -16t^8, 0, -8t^6, -8t^5, 0, 0, 2t^2,\\&-2t, -1, 0))\end{aligned} \tag{47}$$

Note that at $t = 1$, Eq. (46) is recovered.

13.6 Case $[\alpha_1, \alpha_3, \alpha_7, \alpha_9, \alpha_{11}, \alpha_{13}, \alpha_{17}, \alpha_{19}] = [1, 0, -1, 1, 0, 0, 1, 0]$

$$\begin{aligned}&\arctan\left\{t\left(\frac{2t^2-2t+1}{4t^4-2t^3+t-1}\right)\right\}\\&= \frac{1}{1048576t^{39}}P(1, 1048576t^{40}, 40, (524288t^{38}, -524288t^{37},\\&262144t^{36}, 0, 524288t^{34}, 131072t^{33}, -65536t^{32}, 0,\\&32768t^{30}, 131072t^{29}, 16384t^{28}, 0, -8192t^{26},\\&8192t^{25}, 16384t^{24}, 0, 2048t^{22}, -2048t^{21},\\&1024t^{20}, 0, -512t^{18}, 512t^{17}, -256t^{16}, 0,\\&-512t^{14}, -128t^{13}, 64t^{12}, 0, -32t^{10},\\&-128t^9, -16t^8, 0, 8t^6, -8t^5,\\&-16t^4, 0, -2t^2, 2t, -1, 0))\end{aligned} \tag{48}$$

13.7 Case $[\alpha_1,\alpha_3,\alpha_7,\alpha_9,\alpha_{11},\alpha_{13},\alpha_{17},\alpha_{19}]=[0,1,0,0,1,-1,0,1]$

$$\begin{aligned}&\arctan\left\{t\left(\frac{2t^2+2t+1}{4t^4+2t^3-t-1}\right)\right\}\\&=\frac{1}{1048576t^{39}}P(1,1048576t^{40},40,(524288t^{38},524288t^{37},\\&262144t^{36},0,524288t^{34},-131072t^{33},-65536t^{32},0,\\&32768t^{30},-131072t^{29},16384t^{28},0,-8192t^{26},\\&-8192t^{25},16384t^{24},0,2048t^{22},2048t^{21},\\&1024t^{20},0,-512t^{18},-512t^{17},-256t^{16},0,\\&-512t^{14},128t^{13},64t^{12},0,-32t^{10},\\&128t^{9},-16t^{8},0,8t^{6},8t^{5},-16t^{4},0,-2t^{2},-2t,-1,0))\end{aligned}\tag{49}$$

13.8 Case $[\alpha_1,\alpha_3,\alpha_7,\alpha_9,\alpha_{11},\alpha_{13},\alpha_{17},\alpha_{19}]=[0,1,0,0,-1,-1,0,-1]$

$$\begin{aligned}&\sqrt{5}\arctan\left\{t\sqrt{5}\left(\frac{2t^2-1}{4t^4+2t^3-2t^2+t+1}\right)\right\}\\&=\frac{5}{1048576t^{39}}P(1,1048576t^{40},40,(524288t^{38},-524288t^{37},\\&-262144t^{36},0,0,-131072t^{33},65536t^{32},0,32768t^{30},0,\\&16384t^{28},0,8192t^{26},-8192t^{25},0,0,-2048t^{22},\\&-2048t^{21},1024t^{20},0,-512t^{18},512t^{17},\\&256t^{16},0,0,128t^{13},-64t^{12},0,-32t^{10},0,\\&-16t^{8},0,-8t^{6},8t^{5},0,0,2t^{2},2t,-1,0))\end{aligned}\tag{50}$$

14 $n=24$: Base t^{24} formulas

Choosing $n=24$ in Eq. (2), we find

$$\begin{aligned}K&=\alpha_1\mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{i\pi}{24}\right)\right]+\alpha_5\mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{5i\pi}{24}\right)\right]\\&+\alpha_7\mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{7i\pi}{24}\right)\right]\alpha_{11}\mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{11i\pi}{24}\right)\right]\\&+\alpha_{13}\mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{13i\pi}{24}\right)\right]+\alpha_{17}\mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{17i\pi}{24}\right)\right]\\&+\alpha_{19}\mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{19i\pi}{24}\right)\right]+\alpha_{23}\mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{23i\pi}{24}\right)\right]\end{aligned}\tag{51}$$

$$\begin{aligned}=\sum_{r\geq1}\Bigg\{\frac{1}{\sqrt{t^r}r}\Bigg[&\alpha_1\sin\left(\frac{r\pi}{24}\right)+\alpha_5\sin\left(\frac{5r\pi}{24}\right)+\alpha_7\sin\left(\frac{7r\pi}{24}\right)\\&+\alpha_{11}\sin\left(\frac{11r\pi}{24}\right)+\alpha_{13}\sin\left(\frac{13r\pi}{24}\right)+\alpha_{17}\sin\left(\frac{17r\pi}{24}\right)\\&+\alpha_{19}\sin\left(\frac{19r\pi}{24}\right)+\alpha_{23}\sin\left(\frac{23r\pi}{24}\right)\Bigg]\Bigg\}\end{aligned}$$

BBP-type series exist for the following α_j combinations:

$$[\alpha_1,\alpha_5,\alpha_7,\alpha_{11},\alpha_{13},\alpha_{17},\alpha_{19},\alpha_{23}] = \begin{cases} [1,1,1,1,-1,-1,-1,-1] \\ [1,1,-1,-1,1,1,-1,-1] \\ [1,-1,1,-1,1,-1,1,-1] \\ [1,-1,-1,1,-1,1,1,-1] \end{cases} \tag{52}$$

14.1 Case $[\alpha_1,\alpha_5,\alpha_7,\alpha_{11},\alpha_{13},\alpha_{17},\alpha_{19},\alpha_{23}] = [1,1,1,1,-1,-1,-1,-1]$

Replicates Eq. (25)

14.2 Case $[\alpha_1,\alpha_5,\alpha_7,\alpha_{11},\alpha_{13},\alpha_{17},\alpha_{19},\alpha_{23}] = [1,1,-1,-1,1,1,-1,-1]$

Replicates Eq. (18)

14.3 Case $[\alpha_1,\alpha_5,\alpha_7,\alpha_{11},\alpha_{13},\alpha_{17},\alpha_{19},\alpha_{23}] = [1,-1,1,-1,1,-1,1,-1]$

Replicates Eq. (8)

14.4 Case $[\alpha_1,\alpha_5,\alpha_7,\alpha_{11},\alpha_{13},\alpha_{17},\alpha_{19},\alpha_{23}] = [1,-1,-1,1,-1,1,1,-1]$

Replicates Eq. (26)

15 $n = 30$: Base t^{30} formulas

Choosing $n = 30$ in Eq. (2), we find

$$\begin{aligned} K = {} & \alpha_1 \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{i\pi}{30}\right)\right] + \alpha_7 \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{7i\pi}{30}\right)\right] \\ & + \alpha_{11} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{11i\pi}{30}\right)\right] \alpha_{13} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{13i\pi}{30}\right)\right] \\ & + \alpha_{17} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{17i\pi}{30}\right)\right] + \alpha_{19} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{19i\pi}{30}\right)\right] \\ & + \alpha_{23} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{23i\pi}{30}\right)\right] + \alpha_{29} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{29i\pi}{30}\right)\right] \end{aligned} \tag{53}$$

$$\begin{aligned} = \sum_{r\geq 1} \Bigg\{ \frac{1}{\sqrt{t^r}r} \Bigg[& \alpha_1 \sin\left(\frac{r\pi}{30}\right) + \alpha_7 \sin\left(\frac{7r\pi}{30}\right) + \alpha_{11} \sin\left(\frac{11r\pi}{30}\right) \\ & + \alpha_{13} \sin\left(\frac{13r\pi}{30}\right) + \alpha_{17} \sin\left(\frac{17r\pi}{30}\right) + \alpha_{19} \sin\left(\frac{19r\pi}{30}\right) \\ & + \alpha_{23} \sin\left(\frac{23r\pi}{30}\right) + \alpha_{29} \sin\left(\frac{29r\pi}{30}\right) \Bigg] \Bigg\} \end{aligned}$$

BBP-type series exist for the following α_j combinations:

$$[\alpha_1, \alpha_7, \alpha_{11}, \alpha_{13}, \alpha_{17}, \alpha_{19}, \alpha_{23}, \alpha_{29}] = \begin{cases} [1, -1, -1, -1, 1, 1, 1, -1] \\ [1, 1, -1, 1, -1, 1, -1, -1] \\ [1, 1, -1, -1, -1, -1, 1, 1] \\ [1, -1, -1, 1, 1, -1, -1, 1] \end{cases} \tag{54}$$

15.1 Case $[\alpha_1, \alpha_7, \alpha_{11}, \alpha_{13}, \alpha_{17}, \alpha_{19}, \alpha_{23}, \alpha_{29}] = [1, -1, -1, -1, 1, 1, 1, -1]$

Replicates Eq. (38)

15.2 Case $[\alpha_1, \alpha_7, \alpha_{11}, \alpha_{13}, \alpha_{17}, \alpha_{19}, \alpha_{23}, \alpha_{29}] = [1, 1, -1, 1, -1, 1, -1, -1]$

Replicates Eq. (37)

15.3 Case $[\alpha_1, \alpha_7, \alpha_{11}, \alpha_{13}, \alpha_{17}, \alpha_{19}, \alpha_{23}, \alpha_{29}] = [1, 1, -1, -1, -1, -1, 1, 1]$

$$\begin{aligned}
&\sqrt{5} \arctan \left\{ \sqrt{5t} \left(\frac{t^3 - t^2 + t - 1}{t^4 - 2t^3 + 3t^2 - 2t + 1} \right) \right\} \\
&= \frac{5}{\sqrt{t^{59}}} P(1, t^{30}, 60, (t^{29}, 0, -2t^{28}, 0, 0, 0, t^{26}, 0, -2t^{25}, 0, \\
&\quad - t^{24}, 0, -t^{23}, 0, 0, 0, -t^{21}, 0, -t^{20}, 0, -2t^{19}, 0, t^{18}, 0, 0, 0, -2t^{16}, \\
&\quad 0, t^{15}, 0, -t^{14}, 0, 2t^{13}, 0, 0, 0, -t^{11}, 0, 2t^{10}, 0, t^9, 0, t^8, 0, 0, 0, t^6, \\
&\quad 0, t^5, 0, 2t^4, 0, -t^3, 0, 0, 0, 2t, 0, -1, 0))
\end{aligned} \tag{55}$$

15.4 Case $[\alpha_1, \alpha_7, \alpha_{11}, \alpha_{13}, \alpha_{17}, \alpha_{19}, \alpha_{23}, \alpha_{29}] = [1, -1, -1, 1, 1, -1, -1, 1]$

$$\begin{aligned}
&\arctan \left\{ \sqrt{t} \left(\frac{t^3 + t^2 - t - 1}{t^4 - t^2 + 1} \right) \right\} \\
&= \frac{1}{\sqrt{t^{59}}} P(1, t^{30}, 60, (t^{29}, 0, 2t^{28}, 0, -4t^{27}, 0, -t^{26}, 0, -2t^{25}, \\
&\quad 0, -t^{24}, 0, t^{23}, 0, -8t^{22}, 0, t^{21}, 0, -t^{20}, 0, -2t^{19}, 0, -t^{18}, 0, -4t^{17}, \\
&\quad 0, 2t^{16}, 0, t^{15}, 0, -t^{14}, 0, -2t^{13}, 0, 4t^{12}, 0, t^{11}, 0, 2t^{10}, 0, t^9, \\
&\quad 0, -t^8, 0, 8t^7, 0, -t^6, 0, t^5, 0, 2t^4, 0, t^3, 0, 4t^2, 0, -2t, 0, -1, 0))
\end{aligned} \tag{56}$$

16 $n = 40$: Base t^{40} formulas

Choosing $n = 40$ in Eq. (2), we find

$$\begin{aligned}
K &= \alpha_1 \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{i\pi}{40}\right)\right] + \alpha_3 \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{3i\pi}{40}\right)\right] \\
&+ \alpha_7 \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{7i\pi}{40}\right)\right] + \alpha_9 \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{9i\pi}{40}\right)\right] \\
&+ \alpha_{11} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{11i\pi}{40}\right)\right] + \alpha_{13} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{13i\pi}{40}\right)\right] \\
&+ \alpha_{17} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{17i\pi}{40}\right)\right] + \alpha_{19} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{19i\pi}{40}\right)\right] \\
&+ \alpha_{21} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{21i\pi}{40}\right)\right] + \alpha_{23} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{23i\pi}{40}\right)\right] \\
&+ \alpha_{27} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{27i\pi}{40}\right)\right] + \alpha_{29} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{29i\pi}{40}\right)\right] \\
&+ \alpha_{31} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{31i\pi}{40}\right)\right] + \alpha_{33} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{33i\pi}{40}\right)\right] \\
&+ \alpha_{37} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{37i\pi}{40}\right)\right] + \alpha_{39} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{39i\pi}{40}\right)\right]
\end{aligned} \tag{57}$$

$$\begin{aligned}
&= \sum_{r\geq 1}\left\{\frac{1}{\sqrt{t^r}r}\left[\alpha_1\sin\left(\frac{r\pi}{40}\right) + \alpha_3\sin\left(\frac{3r\pi}{40}\right) + \alpha_7\sin\left(\frac{7r\pi}{40}\right)\right.\right. \\
&\quad + \alpha_9\sin\left(\frac{9r\pi}{40}\right) + \alpha_{11}\sin\left(\frac{11r\pi}{40}\right) + \alpha_{13}\sin\left(\frac{13r\pi}{40}\right) + \alpha_{17}\sin\left(\frac{17r\pi}{40}\right) \\
&\quad + \alpha_{19}\sin\left(\frac{19r\pi}{40}\right) + \alpha_{21}\sin\left(\frac{21r\pi}{40}\right) + \alpha_{23}\sin\left(\frac{23r\pi}{40}\right) + \alpha_{27}\sin\left(\frac{27r\pi}{40}\right) \\
&\quad + \alpha_{29}\sin\left(\frac{29r\pi}{40}\right) + \alpha_{31}\sin\left(\frac{31r\pi}{40}\right) + \alpha_{33}\sin\left(\frac{33r\pi}{40}\right) \\
&\quad\quad \left.\left. + \alpha_{37}\sin\left(\frac{37r\pi}{40}\right) + \alpha_{39}\sin\left(\frac{39r\pi}{40}\right)\right]\right\}
\end{aligned}$$

BBP-type series exist for the following α_j combinations:

$$[\alpha_1, \alpha_3, \alpha_7, \alpha_9, \alpha_{11}, \alpha_{13}, \alpha_{17}, \alpha_{19}, \alpha_{21}, \alpha_{23}, \alpha_{27}, \alpha_{29}, \alpha_{31}, \alpha_{33}, \alpha_{37}, \alpha_{39}] = \begin{cases} [1,1,1,1,-1,-1,-1,-1,1,1,1,1,-1,-1,-1,-1] \\ [1,1,-1,1,1,-1,1,1,-1,-1,1,-1,-1,1,-1,-1] \\ [1,-1,1,1,1,1,-1,1,-1,1,-1,-1,-1,-1,1,-1] \\ [1,-1,-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,-1] \end{cases} \tag{58}$$

16.1 Case $[\alpha_1, \alpha_3, \alpha_7, \alpha_9, \alpha_{11}, \alpha_{13}, \alpha_{17}, \alpha_{19}, \alpha_{21}, \alpha_{23}, \alpha_{27}, \alpha_{29}, \alpha_{31}, \alpha_{33}, \alpha_{37}, \alpha_{39}]$ $= [1,1,1,1,-1,-1,-1,-1,1,1,1,1,-1,-1,-1,-1]$

Replicates Eq. (21)

16.2 Case $[\alpha_1, \alpha_3, \alpha_7, \alpha_9, \alpha_{11}, \alpha_{13}, \alpha_{17}, \alpha_{19}, \alpha_{21}, \alpha_{23}, \alpha_{27}, \alpha_{29}, \alpha_{31}, \alpha_{33}, \alpha_{37}, \alpha_{39}]$ $= [1, 1, -1, 1, 1, -1, 1, 1, -1, -1, 1, -1, -1, 1, -1, -1]$

Replicates Eq. (43)

16.3 Case $[\alpha_1, \alpha_3, \alpha_7, \alpha_9, \alpha_{11}, \alpha_{13}, \alpha_{17}, \alpha_{19}, \alpha_{21}, \alpha_{23}, \alpha_{27}, \alpha_{29}, \alpha_{31}, \alpha_{33}, \alpha_{37}, \alpha_{39}]$ $= [1, -1, 1, 1, 1, 1, -1, 1, -1, 1, -1, -1, -1, -1, 1, -1]$

Replicates Eq. (44)

16.4 Case $[\alpha_1, \alpha_3, \alpha_7, \alpha_9, \alpha_{11}, \alpha_{13}, \alpha_{17}, \alpha_{19}, \alpha_{21}, \alpha_{23}, \alpha_{27}, \alpha_{29}, \alpha_{31}, \alpha_{33}, \alpha_{37}, \alpha_{39}]$ $= [1, -1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, -1, 1, 1, -1]$

Replicates Eq. (22)

17 $n = 60$: Base t^{60} formulas

Choosing $n = 60$ in Eq. (2), we find

$$\begin{aligned}
K = {} & \alpha_1 \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{i\pi}{60}\right)\right] + \alpha_7 \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{7i\pi}{60}\right)\right] \\
& + \alpha_{11} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{11i\pi}{60}\right)\right] + \alpha_{13} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{13i\pi}{60}\right)\right] \\
& + \alpha_{17} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{17i\pi}{60}\right)\right] + \alpha_{19} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{19i\pi}{60}\right)\right] \\
& + \alpha_{23} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{23i\pi}{60}\right)\right] + \alpha_{29} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{29i\pi}{60}\right)\right] \\
& + \alpha_{31} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{31i\pi}{60}\right)\right] + \alpha_{37} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{37i\pi}{60}\right)\right] \\
& + \alpha_{41} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{41i\pi}{60}\right)\right] + \alpha_{43} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{43i\pi}{60}\right)\right] \\
& + \alpha_{47} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{47i\pi}{60}\right)\right] + \alpha_{49} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{49i\pi}{60}\right)\right] \\
& + \alpha_{53} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{53i\pi}{60}\right)\right] + \alpha_{59} \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{t}}\exp\left(\frac{59i\pi}{60}\right)\right] \\
= {} & \sum_{r\geq 1}\left\{\frac{1}{\sqrt{t^r}r}\left[\alpha_1 \sin\left(\frac{r\pi}{60}\right) + \alpha_7 \sin\left(\frac{7r\pi}{60}\right) + \alpha_{11} \sin\left(\frac{11r\pi}{60}\right)\right.\right. \\
& + \alpha_{13} \sin\left(\frac{13r\pi}{60}\right) + \alpha_{17} \sin\left(\frac{17r\pi}{60}\right) + \alpha_{19} \sin\left(\frac{19r\pi}{60}\right) + \alpha_{23} \sin\left(\frac{23r\pi}{60}\right) \\
& + \alpha_{29} \sin\left(\frac{29r\pi}{60}\right) + \alpha_{31} \sin\left(\frac{31r\pi}{60}\right) + \alpha_{37} \sin\left(\frac{37r\pi}{60}\right) + \alpha_{41} \sin\left(\frac{41r\pi}{60}\right) \\
& + \alpha_{43} \sin\left(\frac{43r\pi}{60}\right) + \alpha_{47} \sin\left(\frac{47r\pi}{60}\right) + \alpha_{49} \sin\left(\frac{49r\pi}{60}\right) \\
& \left.\left. + \alpha_{53} \sin\left(\frac{53r\pi}{60}\right) + \alpha_{59} \sin\left(\frac{59r\pi}{60}\right)\right]\right\}
\end{aligned} \tag{59}$$

BBP-type series exist for the following α_j combinations:

$$[\alpha_1,\alpha_7,\alpha_{11},\alpha_{13},\alpha_{17},\alpha_{19},\alpha_{23},\alpha_{29},\alpha_{31},\alpha_{37},\alpha_{41},\alpha_{43},\alpha_{47},\alpha_{49},\alpha_{53},\alpha_{59}] = \left\{\begin{array}{l} [1,1,1,1,-1,1,1,-1,-1,1,1,-1,1,1,1,1] \\ [1,1,1,-1,-1,-1,-1,1,1,-1,-1,-1,-1,1,1,1] \\ [1,1,-1,1,-1,1,-1,-1,1,1,-1,1,-1,1,-1,-1] \\ [1,1,-1,-1,-1,-1,1,1,-1,-1,1,1,1,1,-1,-1] \\ [1,-1,1,1,1,-1,1,1,1,1,-1,1,1,1,-1,1] \\ [1,-1,1,-1,1,1,-1,-1,-1,-1,1,1,-1,1,-1,1] \\ [1,-1,-1,1,1,-1,-1,1,-1,1,1,-1,-1,1,1,-1] \\ [1,-1,-1,-1,1,1,1,-1,1,-1,-1,-1,1,1,1,-1] \end{array}\right. \tag{60}$$

17.1 Case $[\alpha_1,\alpha_7,\alpha_{11},\alpha_{13},\alpha_{17},\alpha_{19},\alpha_{23},\alpha_{29},\alpha_{31},\alpha_{37},\alpha_{41},\alpha_{43},\alpha_{47},\alpha_{49},\alpha_{53},\alpha_{59}] = [1,1,1,1,-1,1,1,-1,-1,1,1,-1,1,1,1,1]$

$$\begin{aligned} &-\sqrt{5}\arctan\left\{\sqrt{10t}\left(\frac{1-t-2t^2+2t^5+t^6-t^7}{1-5t-2t^2+5t^3+3t^4+5t^5-2t^6-5t^7+t^8}\right)\right\} \\ &\qquad +\pi\sqrt{5}\sum_{j=2}^{5}\delta_{tj}+2\pi\sqrt{5}\delta_{t1} \\ &=\frac{5\sqrt{2}}{\sqrt{t^{119}}}P(1,t^{60},120,(t^{59},0,2t^{58},0,0,0,t^{56},0,-2t^{55}, \\ &0,t^{54},0,t^{53},0,0,0,-t^{51},0,t^{50},0,2t^{49},0,t^{48},0,0,0,2t^{46},0, \\ &-t^{45},0,-t^{44},0,2t^{43},0,0,0,t^{41},0,2t^{40},0,t^{39},0,-t^{38},0,0, \\ &0,t^{36},0,t^{35},0,-2t^{34},0,t^{33},0,0,0,2t^{31},0,t^{30},0,-t^{29},0, \\ &-2t^{28},0,0,0,-t^{26},0,2t^{25},0,-t^{24},0,-t^{23},0,0,0,t^{21},0,-t^{20}, \\ &0,-2t^{19},0,-t^{18},0,0,0,-2t^{16},0,t^{15},0,t^{14},0,-2t^{13},0,0, \\ &0,-t^{11},0,-2t^{10},0,-t^9,0,t^8,0,0,0,-t^6,0,-t^5,0,2t^4,0,-t^3, \\ &0,0,0,-2t,0,-1,0)) \end{aligned} \tag{61}$$

17.2 Case $[\alpha_1,\alpha_7,\alpha_{11},\alpha_{13},\alpha_{17},\alpha_{19},\alpha_{23},\alpha_{29},\alpha_{31},\alpha_{37},\alpha_{41},\alpha_{43},\alpha_{47},\alpha_{49},\alpha_{53},\alpha_{59}] = [1,1,1,-1,-1,-1,-1,1,1,-1,-1,-1,-1,1,1,1]$

$$\begin{aligned} &\sqrt{3}\arctan\left\{\sqrt{6t}\left(\frac{-1+t-2t^3+2t^4-t^6+t^7}{1-3t+2t^2+3t^3-5t^4+3t^5+2t^6-3t^7+t^8}\right)\right\} \\ &=\frac{3\sqrt{2}}{\sqrt{t^{119}}}P(1,t^{60},120,(t^{59},0,0,0,-4t^{57},0,t^{56},0,0,0,t^{54},0,-t^{53},0,0,0, \\ &-t^{51},0,-t^{50},0,0,0,-t^{48},0,-4t^{47},0,0,0,t^{45},0,t^{44},0,0,0,-4t^{42},0, \\ &-t^{41},0,0,0,-t^{39},0,-t^{38},0,0,0,-t^{36},0,t^{35},0,0,0,t^{33},0,-4t^{32},0,0, \\ &0,t^{30},0,-t^{29},0,0,0,4t^{27},0,-t^{26},0,0,0,-t^{24},0,t^{23},0,0,0,t^{21},0,t^{20},0, \\ &0,0,t^{18},0,4t^{17},0,0,0,-t^{15},0,-t^{14},0,0,0,4t^{12},0,t^{11},0,0,0,t^9,0,t^8, \\ &0,0,0,t^6,0,-t^5,0,0,0,-t^3,0,4t^2,0,0,0,-1,0)) \end{aligned} \tag{62}$$

17.3 Case $[\alpha_1, \alpha_7, \alpha_{11}, \alpha_{13}, \alpha_{17}, \alpha_{19}, \alpha_{23}, \alpha_{29}, \alpha_{31}, \alpha_{37}, \alpha_{41}, \alpha_{43}, \alpha_{47}, \alpha_{49}, \alpha_{53}, \alpha_{59}] = [1, 1, -1, 1, -1, 1, -1, -1, 1, 1, -1, 1, -1, 1, -1, -1]$

Replicates Eq. (37)

17.4 Case $[\alpha_1, \alpha_7, \alpha_{11}, \alpha_{13}, \alpha_{17}, \alpha_{19}, \alpha_{23}, \alpha_{29}, \alpha_{31}, \alpha_{37}, \alpha_{41}, \alpha_{43}, \alpha_{47}, \alpha_{49}, \alpha_{53}, \alpha_{59}] = [1, 1, -1, -1, -1, -1, 1, 1, -1, -1, 1, 1, 1, 1, -1, -1]$

Replicates Eq. (55)

17.5 Case $[\alpha_1, \alpha_7, \alpha_{11}, \alpha_{13}, \alpha_{17}, \alpha_{19}, \alpha_{23}, \alpha_{29}, \alpha_{31}, \alpha_{37}, \alpha_{41}, \alpha_{43}, \alpha_{47}, \alpha_{49}, \alpha_{53}, \alpha_{59}] = [1, -1, 1, 1, 1, -1, 1, 1, 1, 1, -1, 1, 1, 1, -1, 1]$

$$\begin{aligned}
&-\sqrt{15}\arctan\left\{\sqrt{30t}\left(\frac{1-5t+8t^2-8t^3+8t^4-8t^5+5t^6-t^7}{1-15t+38t^2-45t^3+43t^4-45t^5+38t^6-15t^7+t^8}\right)\right\}\\
&\qquad+\pi\sqrt{15}\sum_{j=2}^{12}\delta_{tj}+2\pi\sqrt{15}\delta_{t1}\\
&=\frac{15\sqrt{2}}{\sqrt{t^{119}}}P(1,t^{60},120,(t^{59},0,0,0,0,0,-t^{56},0,0,0,t^{54},0,t^{53},0,0,0,t^{51},0,\\
&-t^{50},0,0,0,t^{48},0,0,0,0,0,t^{45},0,t^{44},0,0,0,0,0,t^{41},0,0,0,-t^{39},0,t^{38},0,\\
&0,0,t^{36},0,t^{35},0,0,0,-t^{33},0,0,0,0,0,t^{30},0,-t^{29},0,0,0,0,0,t^{26},0,0,0,\\
&-t^{24},0,-t^{23},0,0,0,-t^{21},0,t^{20},0,0,0,-t^{18},0,0,0,0,0,-t^{15},0,-t^{14},\\
&0,0,0,0,0,-t^{11},0,0,0,t^9,0,-t^8,0,0,0,-t^6,0,-t^5,0,0,0,t^3,\\
&0,0,0,0,0,-1,0))
\end{aligned} \tag{63}$$

17.6 Case $[\alpha_1, \alpha_7, \alpha_{11}, \alpha_{13}, \alpha_{17}, \alpha_{19}, \alpha_{23}, \alpha_{29}, \alpha_{31}, \alpha_{37}, \alpha_{41}, \alpha_{43}, \alpha_{47}, \alpha_{49}, \alpha_{53}, \alpha_{59}] = [1, -1, 1, -1, 1, 1, -1, -1, -1, -1, 1, 1, -1, 1, -1, 1]$

$$\begin{aligned}
&-\arctan\left\{\sqrt{2t}\left(\frac{1-t+2t^2-2t^3+2t^4-2t^5+t^6-t^7}{1-t+2t^2-3t^3+3t^4-3t^5+2t^6-t^7+t^8}\right)\right\}\\
&=\frac{\sqrt{2}}{\sqrt{t^{119}}}P(1,t^{60},120,(t^{59},0,-2t^{58},0,4t^{57},0,-t^{56},0,-2t^{55},0,t^{54},0,-t^{53},\\
&0,-8t^{52},0,t^{51},0,t^{50},0,2t^{49},0,-t^{48},0,-4t^{47},0,-2t^{46},0,-t^{45},0,-t^{44},0,\\
&-2t^{43},0,-4t^{42},0,-t^{41},0,2t^{40},0,t^{39},0,t^{38},0,-8t^{37},0,-t^{36},0,t^{35},0,\\
&-2t^{34},0,-t^{33},0,4t^{32},0,-2t^{31},0,t^{30},0,-t^{29},0,2t^{28},0,-4t^{27},0,t^{26},0,\\
&2t^{25},0,-t^{24},0,t^{23},0,8t^{22},0,-t^{21},0,-t^{20},0,-2t^{19},0,t^{18},0,4t^{17},0,2t^{16},\\
&0,t^{15},0,t^{14},0,2t^{13},0,4t^{12},0,t^{11},0,-2t^{10},0,-t^9,0,-t^8,0,8t^7,0,t^6,0,\\
&-t^5,0,2t^4,0,t^3,0,-4t^2,0,2t,0,-1,0))
\end{aligned} \tag{64}$$

17.7 Case $[\alpha_1, \alpha_7, \alpha_{11}, \alpha_{13}, \alpha_{17}, \alpha_{19}, \alpha_{23}, \alpha_{29}, \alpha_{31}, \alpha_{37}, \alpha_{41}, \alpha_{43}, \alpha_{47}, \alpha_{49}, \alpha_{53}, \alpha_{59}] = [1, -1, -1, 1, 1, -1, -1, 1, -1, 1, 1, -1, -1, 1, 1, -1]$

Replicates Eq. (56)

17.8 Case $[\alpha_1, \alpha_7, \alpha_{11}, \alpha_{13}, \alpha_{17}, \alpha_{19}, \alpha_{23}, \alpha_{29}, \alpha_{31}, \alpha_{37}, \alpha_{41}, \alpha_{43}, \alpha_{47}, \alpha_{49}, \alpha_{53}, \alpha_{59}]$ $= [1, -1, -1, -1, 1, 1, 1, -1, 1, -1, -1, -1, 1, 1, 1, -1]$

Replicates Eq. (38).

18 Conclusion

We have derived numerous new explicit digit extraction BBP-type formulas for the arctangents of real numbers in general bases, t, without doing computer searches. The high point of this work is the discovery, for the first time, of a binary BBP-type formula for $\pi\sqrt{5}$.

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