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Relations on Jacobsthal numbers

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Abstract: Relations between Jacobsthal numbers, prime Jacobsthal numbers and Fibonacci Jacobsthal numbers are found out.
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1 Introduction

In this paper, we determine a relation between Jacobsthal number and prime Jacobsthal number for twin prime numbers, and an inequality is found between Fibonacci Jacobsthal numbers.

Jacobsthal number is defined as [1, 2, 3]

$$J_n = \frac{2^n - (-1)^n}{3}.$$

The generalised form of n-th order $(n \geq 0)$ Jacobsthal number, Prime Jacobsthal number and Fibonacci Jacobsthal number [1] are defined as

$$J_n^s = \frac{s^n - (-1^n)}{s+1} \tag{1}$$

$$JP_n^s = \frac{p_s^n - (-1^n)}{p_{s+1}} \tag{2}$$

where p_i is the i-th prime number $(p_0 = 2, p_1 = 3, ...)$

$$JF_n^s = \frac{f_s^n - (-1^n)}{f_{s+1}} \tag{3}$$

where f_i is the i-th Fibonacci number $(f_0 = 0, f_1 = 1, f_2 = 1, f_3 = 2, f_4 = 3, ...)$

Theorem 1. *For all natural number m, such that $2m+1$ and $2m+3$ both are prime numbers,*

$$(2m+3)JP_s^n = (m+1)\sum_{x=0}^{n-1} C_x^n 2^{n-x} J_{n-x}^m$$

Theorem 2.

$$(2f_{s+1}+f_s)JF_n^{s+2} > f_{s+1}JF_n^s$$

2 Proof of Theorem 1

We know that twin prime numbers can be represented as $p_s = 2m+1$ and $p_{s+1} = 2m+3$. So we have

$$JP_n^s = \frac{p_s^n - (-1^n)}{p_{s+1}} = \frac{(2m+1)^n - (-1)^n}{2m+3} \tag{4}$$

By binomial expansion for natural number n,

$$(2m+3)JP_n^s = C_0^n(2m)^n + C_1^n(2m)^{n-1}(1) + C_2^n(2m)^{n-2}1^2 + ...+ \\ C_{n-1}^n(2m)^{n-(n-1)} + C_n^n 1^n - (-1)^n$$

$$(2m+3)JP_n^s = 2^n m^n + C_1^n 2^{n-1}m^{n-1} + C_2^n 2^{n-2}m^{n-2} + ...+ \\ C_{n-1}^n 2^{n-(n-1)}m^{n-(n-1)} + 1 - (-1)^n.$$

Adding and subtracting the following term on right hand side of above equation,

$$2^n(-1)^n + C_1^n(2^{n-1}(-1)^{n-1}) + C_2^n(2^{n-2}(-1)^{n-2}) + ... + C_{n-1}^n(2^{n-(n-1)}(-1)^{n-(n-1)}).$$

By grouping the corresponding terms, we get

$$(2m+3)JP_n^s = 2^n(m^n - (-1)^n) + C_1^n 2^{n-1}(m^{n-1} - (-1)^{n-1}) + C_2^n 2^{n-2}(m^{n-2} - (-1)^{n-2}) + ... \\ +C_{n-1}^n 2^{n-(n-1)}(m^{n-(n-1)} - (-1)^{n-(n-1)}) + 2^n(-1)^n + C_1^n 2^{n-1}(-1)^{n-1} \\ +C_2^n 2^{n-2}(-1)^{n-2} + ... + C_{n-1}^n 2^{n-(n-1)}(-1)^{n-(n-1)} + 1 - (-1)^n$$

$$(2m+3)JP_n^s = 2^n(m^n - (-1)^n) + C_1^n 2^{n-1}(m^{n-1} - (-1)^{n-1}) + C_2^n 2^{n-2}(m^{n-2} - (-1)^{n-2}) + ... \\ +C_{n-1}^n 2^{n-(n-1)}(m^{n-(n-1)} - (-1)^{n-(n-1)}) + (-2)^n + C_1^n(-2)^{n-1} \\ +C_2^n(-2)^{n-2} + ... + C_{n-1}^n(-2)^{n-(n-1)} + 1 - (-1)^n$$

We know that

$$(-1)^n = (-2+1)^n = (-2)^n + C_1^n(-2)^{n-1} + C_2^n(-2)^{n-2} + ... + C_{n-1}^n(-2)^{n-(n-1)} + 1.$$

Therefore,

$$(2m+3)JP_n^s = 2^n(m^n - (-1)^n) + C_1^n 2^{n-1}(m^{n-1} - (-1)^{n-1}) + C_2^n 2^{n-2}(m^{n-2} - (-1)^{n-2}) + ...+ \\ C_{n-1}^n 2^{n-(n-1)}(m^{n-(n-1)} - (-1)^{n-(n-1)}) + (-1)^n - (-1)^n.$$

Dividing both sides by $(m+1)$, we get

$$\left(\frac{2m+3}{m+1}\right) JP_n^s = 2^n\left(\frac{m^n-(-1)^n}{m+1}\right) + C_1^n 2^{n-1}\left(\frac{m^{n-1}-(-1)^{n-1}}{m+1}\right)$$

$$+C_2^n 2^{n-2}\left(\frac{m^{n-2}-(-1)^{n-2}}{m+1}\right) ... + C_{n-1}^n 2^{n-(n-1)}\left(\frac{m^{n-(n-1)}-(-1)^{n-(n-1)}}{m+1}\right)$$

$$\left(\frac{2m+3}{m+1}\right) JP_n^s = 2^n J_n^m + C_1^n 2^{n-1} J_{n-1}^m + C_2^n 2^{n-2} J_{n-2}^m + ... + C_{n-1}^n J_{n-(n-1)}^m$$

$$\left(\frac{2m+3}{m+1}\right) JP_n^s = \sum_{x=0}^{n-1} C_x^n 2^{n-x} J_{n-x}^m.$$

This implies that

$$(2m+3)JP_s^n = (m+1)\sum_{x=0}^{n-1} C_x^n 2^{n-x} J_{n-x}^m.$$

This proves the theorem.

3 Proof of Theorem 2

For particular values of n and s, $J_n^{s+2} < J_n^s$. So we determine this generalised inequality.

$$JF_n^{s+2} = \frac{f_{s+2}^n - (-1)^n}{f_{s+3}}$$

From the definition of Fibonacci number, $f_{s+2} = f_s + f_{s+1}$ and $f_{s+3} = f_s + 2f_{s+1}$. Therefore, from binomial expansion

$$(2f_{s+1} + f_s)JF_n^{s+2} = f_s^n + C_1^n f_s^{n-1} f_{s+1} + ... + f_{s+1}^n - (-1)^n$$

this implies

$$(2f_{s+1} + f_s)JF_n^{s+2} > f_{s+1}\left(\frac{f_s^n - (-1)^n}{f_{s+1}}\right)$$

Hence,

$$(2f_{s+1} + f_s)JF_n^{s+2} > f_{s+1} JF_n^s$$

This proves the theorem.

References

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