Notes on Number Theory and Discrete Mathematics
Vol. 19, 2013, No. 3, 12–14

Pulsating Fibonacci sequence

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Abstract: A new type Fibonacci sequence is introduced and explicit formulas for the form of its membres is formulated and proved.
Keywords: Fibonacci sequence.
AMS Classification: 11B39.

To my colleague and friend Prof. Tony Shannon
for his 75^{th} birthday!

During the last century, a lot of extensions and modifications of the Fibonacci sequence were introduced. My friend Tony Shannon and I defined some of them (see, e.g., our book [1]). Here, continuing this direction of research related to Fibonacci sequences, a new type of Fibonacci–like sequence is introduced.

Let a and b be two fixed real numbers. Let us construct the following two sequences

$$\alpha_0 = a,\ \beta_0 = b,$$

$$\alpha_{2k+1} = \beta_{2k+1} = \alpha_{2k} + \beta_{2k},$$

$$\alpha_{2k+2} = \alpha_{2k+1} + \beta_{2k},$$

$$\beta_{2k+2} = \beta_{2k+1} + \alpha_{2k},$$

for the natural numner $k \geq 0$. This pair of sequences we call a Pulsating Fibonacci sequence.

The first values of the two sequences are given in the following Table 1.

Table 1.

k	α_k	$\alpha_k = \beta_k$	β_k
0	a		b
1		$a+b$	
2	$a+2b$		$2a+b$
3		$3a+3b$	
4	$5a+4b$		$4a+5b$
5		$9a+9b$	
6	$13a+14b$		$14a+13b$
7		$27a+27b$	
8	$41a+40b$		$40a+41b$
9		$81a+81b$	
10	$121a+122b$		$122a+121b$
11		$243a+243b$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Theorem. For every natural number $k \geq 0$,

$$\alpha_{2k+1} = \beta_{2k+1} = 3^k a + 3^k b, \tag{1}$$

$$\alpha_{4k} = \frac{3^{2k}+1}{2}a + \frac{3^{2k}-1}{2}b, \tag{2}$$

$$\beta_{4k} = \frac{3^{2k}-1}{2}a + \frac{3^{2k}+1}{2}b, \tag{3}$$

$$\alpha_{4k+2} = \frac{3^{2k+1}-1}{2}a + \frac{3^{2k+1}+1}{2}b, \tag{4}$$

$$\beta_{4k+2} = \frac{3^{2k+1}+1}{2}a + \frac{3^{2k+1}-1}{2}b. \tag{5}$$

Proof. Obviously, for $k = 0$ the assertion is valid. Let us assume that for some natural number $k \geq 0$, (1)–(5) are valid. For the natural number $k+1$, first, we check that

$$\alpha_{4k+1} = \beta_{4k+1} = \alpha_{4k} + \beta_{4k} = \frac{3^{2k}+1}{2}a + \frac{3^{2k}-1}{2}b + \frac{3^{2k}-1}{2}a + \frac{3^{2k}+1}{2}b$$

$$= \frac{3^{2k}+1+3^{2k}-1}{2}a + \frac{3^{2k}-1+3^{2k}+1}{2}b = 3^{2k}a + 3^{2k}b.$$

Second, we check that

$$\alpha_{4k+1} = \alpha_{4k} + \beta_{4k-1} = \frac{3^{2k}+1}{2}a + \frac{3^{2k}-1}{2}b + 3^{2k}a + 3^{2k}b$$

$$= \frac{3^{2k}+1+2.3^{2k}}{2}a + \frac{3^{2k}-1+2.3^{2k}}{2}b = \frac{3^{2k+1}+1}{2}a + \frac{3^{2k+1}-1}{2}b.$$

All other eqalities are checked analogously.

For example, if $b = -a$, then the Pulsating Fibonacci sequence has the form:

k	α_k	$\alpha_k = \beta_k$	β_k
0	a		$-a$
1		0	
2	$-a$		a
3		0	
4	a		$-a$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

while, if $b = a$, then the Pulsating Fibonacci sequence has the form:

k	α_k	$\alpha_k = \beta_k$	β_k
0	a		a
1		$2a$	
2	$3a$		$3a$
3		$6a$	
4	$9a$		$9a$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

References

[1] Atanassov K., V. Atanassova, A. Shannon, J. Turner, *New Visual Perspectives on Fibonacci Numbers*. World Scientific, New Jersey, 2002.