Notes on Number Theory and Discrete Mathematics
Vol. 19, 2013, No. 3, 5–11

The Pascal–Fibonacci numbers

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Abstract. The Pascal–Fibonacci (PF) numbers for a given Fibonacci number sum to give the values of that Fibonacci number. Individual PF numbers are members of one of the triangular, tetrahedral or pentagonal series or have factors in common with the pyramidal or other geometric series. For composite numbers, partial sums of PF numbers can yield a factor, while prime Fibonacci numbers are detected with sums of squares.

Keywords: Fibonacci numbers, Binet equation, Pascal Triangle, Triangular numbers, Tetrahedral numbers, Pentagonal numbers, Pyramidal numbers.
AMS Classification: 11B39, 11B50.

1 Introduction

It is well-known that the numbers along the leading diagonals in the Pascal Triangle sum to numbers in the Fibonacci sequence $\{F_n\}$ (with generalizations to higher order recursive sequences and geometric dimensions [7]). This provides a simple way to calculate the Fibonacci numbers without irrationals, as in the Binet equation [5, 6]; that is

$$F_n = 2 + \sum_{i=2}^{\frac{1}{2}(p-1)} \binom{p-i}{i-1}. \tag{1.1}$$

We have called elements of these Pascal–Fibonacci (PF) numbers, each of which is given by

$$N_{p_i} = \binom{p-i}{i-1} \tag{1.2}$$

which are listed in Table 1 for p from 7 to 59 [6]. For example, when $p = 17$ and $i = 4$, the third number in the sum is $N_{P_{17}}(3) = 286$. Similarly, when $p = 43$ and $i = 5$, $N_{P_{43}}(4) = 73815$. Again, when $p = 59$, the last $i = ½(p - 1) = 29$, so the 28[th] number in the sum is $N_{P_{59}}(28) =$ $= 30! / 28!\times 2! = 435$.

p	$\boldsymbol{F_p}$
7	5， 6
11	9， 28,35,15
13	11， 45,84,70,21
17	15， 91,286,495,462,210,36
19	17， 120,455,1001,1287,924,330,45
23	21， 190,969,3060,6188,8008,6435,3003,715,66
29	27， 325,2300,10626,33649,74613,116280,125970,92378,43758,12376,1820,105
31	29， 378,2925,14950,53130,134596,245157,319770,293930,184756,75582,18564,2380,120
37	35， 561,5456,35960,169911,593775,1560780,3108105,4686825,5311735,4457400, 2704156,1144066,319770,54264,4845,171
41	39， 703,7770,58905,324632,1344904,4272048,10518300,20160075,30045015, 34597290,30421755,20058300,9657700,3268760,735471,100947,7315,210
43	41， 780,9139,73815,435897,1947792,6724520,18156204,38567100,64512240, 84672315,86493225,67863915,40116600,17383860,5311735,1081575,134596, 8855,231
47	45， 946,12341,111930,749398,3838380,15380937,48903492,124403620,254186856,417225900, 548354040,573166440,471435600,300540195,145422675,51895935,13123110,2220075, 230230,12650,276
53	51， 1225,18424,194580,1533939,9366819,45379620,177232627,563921995,1471442973, 3159461968,5586853480,8122425444,9669554100,9364199760,7307872110,4537567650, 2203961430,818809200,225792840,44352165,5852925,475020,20475,351
59	57， 1540,26235,316251,2869685,20358520,115775100,536878650,2054455634,6540715896, 17417133617,38910617655,73006209045,114955808528,151532656696,166509721602, 151584480450,113380261800,68923264410,33578000610,12875774670, 3796297200,834451800,131128140,13884156,906192,31465,435

Table 1. Pascal–Fibonacci numbers

In this paper, we examine the details of the structure of these numbers. In particular, the PF numbers are formed by sequential ratios of factorials and therefore have a regular structure. They are composed of primes, the maximum of which is P_{i-1} for F_{P_i}. For instance, the PF numbers associated with F_{19} have the primes {2, 3, 5, 7, 11, 13, 17} in varying proportions. The simple structure makes the position of each along the diagonals significant. For example, all the first numbers equal (p – 2), the second are triangular as are the last, while the third numbers are tetrahedral. The second last numbers, N_i, (i=½(p – 5)), are pentagonal [2, 3].

2 The second PF numbers

The triangular numbers can be represented by

$$T_n = \tfrac{1}{2}n(n + 1). \qquad (2.1)$$

The second PF number of each F_p ($7 \le p \le 59$) are given by Equation (2.1) with $n = p - 4$ (Table 2).

p	$n = p - 4$	$N_2 = \frac{1}{2}n(n+1)$
7	3	6
11	7	28
13	9	45
17	13	91
19	15	120
23	19	190
29	25	325
31	27	378
37	33	561
41	37	703
43	39	780
47	43	946
53	49	1225
59	55	1540
61	57	1653

Table 2. Second PF Numbers

3 The third PF numbers

The tetrahedral numbers in this context [2, 3] are given by

$$H_n = \tfrac{1}{6}n(n+1)(n+2) \tag{3.1}$$

and the third PF numbers fit this series with $n = (p - 6)$ (Table 3).

p	$n = p - 6$	$N_3 = \frac{1}{6}n(n+1)(n+2)$
11	5	35
13	7	84
17	11	286
19	13	455
23	17	969
29	23	2300
31	25	2925
37	31	5456
41	35	7770
43	37	9139
47	41	12341
53	47	18424
59	53	26235
61	55	29260

Table 3. Third PF Numbers

4 The fourth PF numbers

Assuming the pattern for n continues, that is $(p - 2i)$ for n, are the fourth numbers compatible with some geometric number series? Using $n = p - 8$ for the pyramidal numbers [2, 3], that is

$$Q_n = \tfrac{1}{6}n(n+1)(2n+1) \quad (4.1)$$

it is found that these numbers always have a factor in common with the fourth PF numbers. In fact, the factor 5 is common to all fourth PF numbers except F_{19}, F_{29}, F_{59}; that is, when p has a right-end-digit (RED) of 9 and is therefore an element of the Class $\bar{4}_5 \subset Z_5$ [1, 4, 6].

5 Last three numbers

(a) The last numbers are triangular numbers which satisfy Equation (2.1) with $n = ½(p - 1)$. All these numbers are divisible by 3 so that they fall into a special subset of the triangular numbers [2, 3] (Table 4).

p	$n = ½(p - 1)$	N_i (Eq 2.1)	$⅓N_i$
7	3	6	2
11	5	15	5
13	6	21	7
17	8	36	12
19	9	36	15
23	11	66	22
29	14	105	35
31	15	120	40
37	18	171	57
41	20	210	70
43	21	231	77
47	23	276	92
53	26	351	117
59	29	435	145
61	30	465	155

Table 4. $i = ½(p - 3)$

(b) The second last numbers N_i ($i = ½(p - 5)$) always have 5 as a factor. These numbers are always divisible by 5, and at least one of the factors is triangular with n varying from 1 to 22. Moreover, these numbers are pentagonal, given by [2, 3]:

$$D_n = \tfrac{1}{2}n(3n-1) \quad (5.1)$$

with $n = p^2/24$ (Table 5).

p	$n = p^2/24$ *	D_i = ½$n(3n – 1)$	position, i
11	5	35	3
13	7	70	4
17	12	210	6
19	15	330	7
23	22	715	9
29	35	1820	12
31	40	2380	13
37	57	4845	16
41	70	7315	18
43	77	8855	19
47	92	12650	21
53	117	20475	24
59	145	31465	24
61	155	35960	28

Table 5. Pentagonal numbers D_i = ½$n(3n – 1)$ and position i;
*residual of 0.04 for all p is neglected

(c) Third last numbers, N_i, i = ½(p – 7), are always even (except for p = 23 or 41), and all are divisible by 7 (11 numbers from p = 17 to 59). Eight of the numbers also have 8 as a factor.

6 Other numbers, N_i, with i = 4 to 14

The remaining numbers have factors in common with triangular, tetrahedral and other geometric series, with $n = p – 2i$. They generally have a factor common to all, but there are a few exceptions: examples (Table 6):

i	Common factor for i	Exceptions	
		p	Factor of N_i
4	5	29,59	3
5	3	13,19,31,37	7
6	7	41	11
7	5	31,41,47	3
8	3	53	11
9	5	29,59	11
10	11	43	3
11	5	29	7
12	5	31,37	7
13	5	37,53	7
14	5	59	11

Table 6. Remaining numbers, N_i

7 Identifying prime F_p

We have previously [6] shown how the structure of the Fibonacci numbers can help to identify primes. Some alternative complementary methods are outlined in (a) and (b) below.

7.1 From PF numbers

The PF numbers can be partially summed to find a suitable factor (Table 7).

p	F_p	Numbers summed	Factors
19	4181	$S = N_1 + N_2 + N_3 = 17 + 120 + 145 = 37 \times 16$	37×113
37	24157817	$S = N_1 + N_2 = 35 + 561 = 4 \times 149$	$73 \times 149 \times 2221$

Table 7. Partial sums of PF numbers

7.2 Sum of squares

Another method is via the sum of squares [1]:

$$F_p = d^2 + e^2. \tag{7.1}$$

Primes only have one set of (d, e) with no common factors. Generally composites have the same number of sets as their factors. One (d, e) set is given by

$$F_p = \left(F_{\frac{1}{2}(p+1)}\right)^2 + \left(F_{\frac{1}{2}(p-1)}\right)^2 \tag{7.1}$$

with others from various techniques [1]. Examples are displayed in Table 8.

p	F_p	factors	d	e	d, e as F_n	
7	13	-	3	2	F_4	F_3
11	89	-	5	8	F_5	F_6
13	233	-	13	8	F_7	F_6
17	1597	-	21	34	F_8	F_9
19	4181	37,113	55	34	F_{10}	F_9
			41	50		
23	28657	-	89	144	F_{11}	F_{12}
29	514229	-	377	610	F_{14}	F_{15}
31	1346269	557,2417	987	610	F_{16}	F_{15}
			875	762		
37	24157817	73,149,2221	4181	2584	F_{19}	F_{18}
			4909	244		
			3859	3044		

Table 8. Sums of squares

Finally, the interested reader may like to develop generalizations of the Pascal–Fibonacci numbers which can be made with suitable generalizations of Pascal's triangle [8].

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