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Note on some explicit formulae for twin prime counting function

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Abstract: In the paper for any integer $n \geq 5$ the validity of the formula

$$\pi_2(n) = 1 + \left\lfloor \sum_{k=1}^{\lfloor \frac{n+1}{6} \rfloor} \left(\frac{\varphi(36\,k^2 - 1)}{36\,k^2 - 12k} \right)^{k^2} \right\rfloor$$

(where π_2 denotes the twin prime counting function and φ is Euler's totient function) is established. Also for any integer $n \geq 5$ the formula

$$\pi_2(n) = 1 + \left\lfloor \sum_{k=1}^{\lfloor \frac{n+1}{6} \rfloor} \left(\frac{\varphi(36\,k^2 - 1)}{36\,k^2 - 12k} \right)^{6\,k \ln f(k)} \right\rfloor$$

(where f is arbitrary artihmetic function with strictly positive values satisfying the condition $\sum\limits_{k=4}^{\infty} \frac{1}{f(k)} < 1$) is proved.
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Used denotations

$\lfloor\rfloor$ – denotes the floor function, i.e. $\lfloor x \rfloor$ denotes the largest integer that is not greater than the real non-negative number x; π_2 - the twin prime counting function, i.e. for any integer $n \geq 3$, $\pi_2(n)$ denotes the number of couples $(p, p+2)$ such that $p \leq n$ and p and $p+2$ are both primes; φ denotes Euler's totient function, i.e. $\varphi(1) = 1$ and for any integer $n \geq 2$

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p} \right),$$

where $\prod\limits_{p|n}$ means that the product is taken over all prime divisors p of n (it is known that $\varphi(n)$ coincides with the number of all integer $k \le n$ which are relatively prime to n).

1 Introduction

As known (see [1, p. 259]), twin primes are couples of the kind $(p, p+2)$, where p and $p+2$ are both primes. It is clear that (with the only exception being the couple $(3,5)$) all couples of twin primes have the form $(6k-1, 6k+1)$ with an appropriate integer $k \ge 1$. The sequence of all twin primes is: $3, 5, 7, 11, 13, 17, 19, 29, 31, 41, 43, \ldots$. The couples of twin primes are: $(3,5), (5,7), (11,13), (17,19), (29,31), (41,43), \ldots$.

So for any integer $n \ge 5$, for the twin prime counting function π_2, the representation

$$\pi_2(n) = 1 + \sum_{k=1}^{\left\lfloor \frac{n+1}{6} \right\rfloor} \delta_k^* \tag{1}$$

is valid, where:

$$\delta_k^* = \begin{cases} 1, & \text{if } (6k-1, 6k+1) \text{ is a couple of primes} \\ 0, & \text{otherwise.} \end{cases}$$

Using the mentioned above, in year 2001, the author (in [2]) proposed (for the first time) fourteen different formulae for the twin prime counting function π_2. One of them is the following:

$$\pi_2(n) = 1 + \sum_{k=1}^{\left\lfloor \frac{n+1}{6} \right\rfloor} \left\lfloor \left(\frac{\varphi(36\,k^2-1)}{36\,k^2-12k} \right) \right\rfloor .$$

Unfortunately in this formula the floor function is applied to each term of the sum which makes it not sufficiently convenient. But it is possible to remedy this disadvantage with the help of the formulae that we propose for the first time in the present paper. In these formulae the floor function is applied to the whole sum.

2 Preliminary result

Let for integer $k \ge 1$ δ_k is introduced by:

$$\delta_k \stackrel{\text{def}}{=} \frac{\varphi(36\,k^2-1)}{36\,k^2-12\,k}. \tag{2}$$

Further we need the following result.

Lemma 1. *$(6k-1, 6k+1)$ is a couple of twin primes if and only if $\delta_k = 1$. Otherwise,*

$$\delta_k \le 1 - \frac{1}{6k}. \tag{3}$$

Proof. Since for any integer $k \geq 1$ $6k-1$ and $6k+1$ are relatively prime, and because of the multiplicativity of the function φ, we have

$$\delta_k = \frac{\varphi((6k-1)(6k+1))}{(6k-2)6k} = \frac{\varphi(6k-1)\varphi(6k+1)}{(6k-2)6k}. \tag{4}$$

Let for some integer $k \geq 1$ the numbers $6k-1$ and $6k+1$ are both primes. Then:

$$\varphi(6k-1) = 6k-2; \varphi(6k+1) = 6k$$

and (4) implies $\delta_k = 1$.

Conversely, let $\delta_k = 1$. Since $\varphi(6k-1) \leq 6k-2$ and $\varphi(6k+1) \leq 6k$, we must have:

$$\varphi(6k-1) = 6k-2; \varphi(6k+1) = 6k.$$

But the last means that the numbers $6k-1$ and $6k+1$ are both primes, i.e. $(6k-1, 6k+1)$ is a twin prime couple.

Otherwise, we have three different cases:

(a) $6k-1$ and $6k+1$ are composite;

(b) $6k-1$ is prime and $6k+1$ is composite;

(c) $6k-1$ is composite and $6k+1$ is prime.

Let (a) hold. Then

$$\varphi(6k-1) \leq 6k-3; \varphi(6k+1) \leq 6k-1$$

and (4) yields

$$\delta_k = \frac{\varphi((6k-1)(6k+1))}{(6k-2)6k} \leq \frac{(6k-1)}{6k} = 1 - \frac{1}{6k}.$$

Therefore, for (a) (3) is valid.

Let (b) hold. Then

$$\varphi(6k-1) = 6k-2; \varphi(6k+1) \leq 6k-1.$$

Hence (4) yields

$$\delta_k \leq \frac{(6k-2)(6k-1)}{(6k-2)6k} = 1 - \frac{1}{6k}.$$

Therefore, for (b) (3) is valid.

Let (c) hold. Then

$$\varphi(6k-1) \leq 6k-3; \varphi(6k+1) = 6k.$$

Hence (4) yields

$$\delta_k \leq \frac{(6k-3)6k}{(6k-2)6k} = \frac{6k-3}{6k-2} < 1 - \frac{1}{6k}.$$

Therefore, for (c) (3) is valid.

The lemma is proved. □

3 Main results

The first main result of the paper is the following.

Theorem 1. *Let f is arbitrary artihmetic function with strictly positive values satisfying the condition*

$$\sum_{k=4}^{\infty} \frac{1}{f(k)} < 1. \tag{5}$$

Then for any integer $n \geq 5$ the formula

$$\pi_2(n) = 1 + \left\lfloor \sum_{k=1}^{\left\lfloor \frac{n+1}{6} \right\rfloor} \left(\frac{\varphi(36\,k^2-1)}{36\,k^2-12k} \right)^{6\,k \ln f(k)} \right\rfloor \tag{6}$$

is valid.

Proof. Let $n \geq 5$ be arbitrarily chosen. We denote

$$R \stackrel{\text{def}}{=} 1 + \sum_{k=1}^{\left\lfloor \frac{n+1}{6} \right\rfloor} \left(\frac{\varphi(36\,k^2-1)}{36\,k^2-12k} \right)^{6\,k \ln f(k)}. \tag{7}$$

Then

$$R = 1 + \sum_{k=1}^{\left\lfloor \frac{n+1}{6} \right\rfloor} \delta_k^{6\,k \ln f(k)} = \sum\nolimits_1 \delta_k^{6\,k \ln f(k)} + \sum\nolimits_2 \delta_k^{6\,k \ln f(k)},$$

where $\sum_1$ is a sum over all integer k satisfying $1 \leq k \leq \left\lfloor \frac{n+1}{6} \right\rfloor$, such that $\delta_k = 1$ and $\sum_2$ is a sum over all integer k such that $\delta_k < 1$ (satisfying $k \leq \left\lfloor \frac{n+1}{6} \right\rfloor$).

Hence

$$R = \pi_2(n) + \sum\nolimits_2 \delta_k^{6\,k \ln f(k)}, \tag{8}$$

since

$$\sum\nolimits_1 \delta_k^{6\,k \ln f(k)} = \pi_2(n) - 1,$$

because of the Lemma.

Therefore, to prove (6) it remains only to establish that, under the conditions of Theorem 1,

$$\sum\nolimits_2 \delta_k^{6\,k \ln f(k)} < 1. \tag{9}$$

Let k is such that $\delta_k < 1$. Since (3) holds, we have

$$\delta_k^{6\,k} \leq (1 - \frac{1}{6k})^{6k} < e^{-1}, \tag{10}$$

where

$$e \stackrel{\text{def}}{=} \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^m = 2.718\ldots$$

and for the monotonously increasing sequence $\{\left(1 - \frac{1}{m}\right)^m\}_{m=2}^{\infty}$ it is fulfilled

$$e^{-1} = \lim_{m \to \infty} \left(1 - \frac{1}{m}\right)^m.$$

Therefore,

$$\sum\nolimits_2 \delta_k^{6\,k\ln f(k)} < \sum_{k=4}^{\lfloor \frac{n+1}{6} \rfloor} e^{-\ln f(k)} = \sum_{k=4}^{\lfloor \frac{n+1}{6} \rfloor} \frac{1}{f(k)} < 1$$

(because of (5)).

Theorem 1 is proved.

□

Let $n \geq 77$. Then $13 \leq \lfloor \frac{n+1}{6} \rfloor$ and (8) yields

$$R = \pi_2(n) + \tilde{\delta}_6 + \tilde{\delta}_8 + \tilde{\delta}_9 + \tilde{\delta}_{11} + \sum\nolimits_2 \delta_k^{6\,k\ln f(k)}, \tag{11}$$

where $\sum_2$ is considered for k satisfying $13 \leq k \leq \frac{n+1}{6}$ and

$$\tilde{\delta}_i \overset{\text{def}}{=} (\delta_i)^{6i\ln f(i)}.$$

Let $f(x) = e^{\frac{x}{6}}$. Then:

$$\delta_4 = \frac{5}{6}; \delta_6 = \frac{12}{17}; \delta_8 = \frac{7}{8}; \delta_9 = \frac{20}{27}; \delta_{11} = \frac{3}{4}$$

and

$$\tilde{\delta}_i = (\delta_i)^{i^2}.$$

The direct computation yields

$$\tilde{\delta}_6 + \tilde{\delta}_8 + \tilde{\delta}_9 + \tilde{\delta}_{11} < 0.06. \tag{12}$$

Using (10), this time we have:

$$\sum\nolimits_2 \delta_k^{6\,k\ln f(k)} < \sum_{k=13}^{\infty} e^{-\ln f(k)} = \sum_{k=13}^{\infty} \frac{1}{f(k)} = \sum_{k=13}^{\infty} e^{-\frac{k}{6}} = \frac{e^{-2}}{e^{\frac{1}{6}} - 1} < 0.75. \tag{13}$$

Since $0.06 + 0.75 < 1$, from (11), (12) and (13) it follows

$$\lfloor R \rfloor = \pi_2(n).$$

Therefore, (11) and the last equality proved (6) for $n \geq 77$ with $f(x) = e^{\frac{x}{6}}$.

But it is a matter of a direct check to see that in this case (6) is valid also for $5 \leq n \leq 76$.

Since $6k\ln f(k) = k^2$ for $f(k) = e^{\frac{k}{6}}$, we proved the second main result of our paper.

Theorem 2. *For any integer $n \geq 5$ the representation*

$$\pi_2(n) = 1 + \left\lfloor \sum_{k=1}^{\lfloor \frac{n+1}{6} \rfloor} \left(\frac{\varphi(36\,k^2 - 1)}{36\,k^2 - 12k} \right)^{k^2} \right\rfloor$$

holds.

References

[1] Ribenboim, P. *The New Book of Prime Number Records* (3rd Edition), Springer-Verlag, New York, 1996.

[2] Vassilev-Missana, M. Some new formulae for the twin prime counting function π_2, *Notes on Number Theory and Discrete Mathematics*, Vol 7, 2001, No. 1, 10–14.