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Two-term Egyptian fractions

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Abstract: We study two-term Egyptian fraction representations of a given rational number. We consider the case of m/n where each prime factor p of n satisfies $p \equiv \pm 1 \pmod{m}$: necessary and sufficient conditions for the existence of proper two-term Egyptian fraction expressions of such m/n are given, together with methods to find these representations. Furthermore, we determine the number of proper two-term Egyptian fraction expressions for $1/m$, $2/m$, $3/m$, $4/m$ and $6/m$.
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1 Introduction

Egyptian fractions were used by Egyptian mathematicians some 4000 years ago to represent rational numbers. A *unit fraction* is a fraction of the form $1/n$ where $n \geqslant 2$. An *Egyptian fraction* is an expression that is a sum of unit fractions

$$\frac{m}{n} = \frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_k} \tag{1}$$

where $0 < m/n < 1$ and $k \geqslant 1$. We say that the Egyptian fraction (1) is *proper* if its unit fractions are distinct. Thus the expression $2/3 = 1/3+1/3$ is improper, whereas, $2/3 = 1/2+1/6$ is proper. Egyptian fractions are awkward for arithmetical calculations; however, such representations has led to intriguing questions in number theory, see Guy [3].

We first dispense with two immediate questions, that of existence and uniqueness. Indeed, every rational number admits a proper Egyptian fraction representation, and secondly, every rational number has infinitely many proper representations. For the proof of existence we present Fibonacci's algorithm for producing proper Egyptian fractions.

Theorem 1. *Let* $0 < m/n < 1$. *Then there is a proper Egyptian fraction*

$$\frac{m}{n} = \frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_k}$$

where $k \geqslant 1$ *and* $2 \leqslant n_1 < n_2 < \cdots < n_k$. *Moreover, there are infinitely many proper Egyptian fractions for* m/n.

Proof. We use induction on m. The result it true when $m = 1$. Now assume that $m > 1$ and that the hypothesis is true for all $0 < M/N < 1$ with $M < m$. We subtract the largest possible unit fraction from m/n. Namely, there is a unique $n_1 \geqslant 2$ such that

$$\frac{1}{n_1} < \frac{m}{n} < \frac{1}{n_1 - 1}. \tag{2}$$

Indeed n_1 is the smallest integer greater than or equal to n/m. Write $M/N = m/n - 1/n_1 = (mn_1 - n)/nn_1$, whence $0 < M/N < 1$. The second inequality in (2) shows that $mn_1 - n < m$, hence $M < m$. By the induction hypothesis there is a proper Egyptian fraction

$$\frac{M}{N} = \frac{1}{n_2} + \cdots + \frac{1}{n_k}$$

where $k \geqslant 2$ and $2 \leqslant n_2 < \cdots < n_k$. It remains to show that $n_1 < n_2$. For a contradiction, suppose that $n_1 \geqslant n_2$. It follows that

$$\frac{1}{n_1 - 1} > \frac{m}{n} = \frac{M}{N} + \frac{1}{n_1} \geqslant \frac{1}{n_2} + \frac{1}{n_1} \geqslant \frac{2}{n_1}$$

which implies that $n_1 < 2$, contradicting $n_1 \geqslant 2$. This completes the induction.

Next, repeated applications of the equality

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)},$$

for instance

$$\frac{1}{3} = \frac{1}{4} + \frac{1}{12} = \frac{1}{4} + \frac{1}{13} + \frac{1}{12 \cdot 13} = \ldots,$$

shows that each unit fraction has infinitely many proper representations, and this implies that any m/n has infinitely many proper representations. □

Example 1. To apply Fibonacci's algorithm for $4/5$ we first find the largest unit fraction smaller than $4/5$; one method is to increase the denominator until it just becomes a multiple of the numerator, obtaining $4/8 = 1/2$. Applying this to $4/5$ and then to $3/10$ we obtain

$$\frac{4}{5} = \frac{1}{2} + \frac{3}{10} = \frac{1}{2} + \frac{1}{4} + \frac{1}{20}.$$

However we note the alternative expression

$$\frac{4}{5} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10}.$$

We do not know the methods used by the Egyptians for producing their fractions. However it seems that they preferred fractions with small denominators. Thus we may ask, given m/n and Egyptian fractions (1) for m/n,

(I) what is the length k of a shortest Egyptian fraction for m/n, and the number of representations of shortest length,

(II) what is the representation for which the last denominator n_k is a minimum.

Question (I) is to specify conditions (C) on m, n, and determine the smallest k such that all rationals m/n satisfying conditions (C) possess an Egyptian fraction representation of length k. It is known that the minimum number of terms to express any $2/n$ as an Egyptian fraction is two, and the minimum number of terms for any $3/n$ is two or three, according as $n \equiv 2$ or $1 \pmod 3$ respectively. However for the fraction m/n, where $m \geqslant 4$, the minimal length of a shortest Egyptian fraction is still unknown. Among the unknown cases, the Erdős-Straus conjecture states that $4/n$ can be expressed as an Egyptian fraction with at most three unit fractions.

Existing studies of minimal length Egyptian fractions for m/n focus on the minimum number k of terms, but not on how many such Egyptian fractions there are, nor on how to find them all, which are in fact very important aspects of Egyptian fractions.

In this paper we focus on two-term Egyptian fractions, that is, the Diophantine equation $1/x + 1/y = m/n$, splitting a rational m/n into two unit fractions. We first state formulae for obtaining all two-term expansions for m/n in Theorems 3 and 4. The number of two-term expansions of $1/n$ and for $2/n$ are given in Theorems 6 and 7. These results set the stage for the main result of this paper: namely, in Theorem 12, we give the number of two-term Egyptian fractions for m/n, under certain conditions on m, n, and we outline methods for finding all such Egyptian fractions. The result is then applied to rationals $3/n$, $4/n$ and $6/n$.

2 The equation $1/x + 1/y = m/n$

A given m/n may not admit a two-term Egyptian fraction. In this section we give a constructive method for finding all two-term Egyptian fractions for a given m/n, or else determining that no such expansions exist. That is, we wish to determine all *positive integer* solutions x, y to the equation $1/x + 1/y = m/n$, given $1 \leqslant m \leqslant n$ and $\gcd(m, n) = 1$. The results in this section are scattered in the literature. However, since the main theorems in Section 3 depend on these results, for convenience we give the proofs.

Denote by $\tau(n)$ the number of positive divisors of n, where $n \geqslant 1$. Then $\tau(1) = 1$, and if $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ is the prime factorization of n, then $\tau(n) = (r_1 + 1)(r_2 + 1) \cdots (r_k + 1)$. Moreover, τ is multiplicative, that is, $\tau(mn) = \tau(m)\tau(n)$ whenever $\gcd(m, n) = 1$.

Theorem 2. *Let $n \geqslant 1$ be given. Consider the equation*

$$1/x + 1/y = 1/n.$$

If (x, y) is a positive solution then there exist positive integers f_1, f_2 such that $f_1 f_2 = n^2$ and $x = n + f_1$, $y = n + f_2$.

Conversely, if f_1, f_2 are positive integers such that $f_1 f_2 = n^2$ then $x = n + f_1$, $y = n + f_2$ is a positive solution.

Hence the number of positive solution pairs (x, y) to the equation is equal to $\tau(n^2)$.

Proof. Suppose that (x, y) is a positive solution. Then $x > n$, $y > n$ and

$$y = \frac{nx}{x - n} = n + \frac{n^2}{x - n}.$$

Hence $(x - n)(y - n) = n^2$. Put $f_1 = x - n$, and $f_2 = y - n$. Then f_1, f_2 are positive, $f_1 f_2 = n^2$, and $x = n + f_1$, $y = n + f_2$.

Conversely, let f_1, f_2 be positive and satisfy $f_1 f_2 = n^2$. It is easy to check that $x = n + f_1$, $y = n + f_2$ is a positive solution to the given equation.

Thus the map $f \mapsto (n + f, n + n^2/f)$ is a bijection from the set of positive divisors of n^2 to the set of positive solutions (x, y). □

The next result can be found in Bartoš [1].

Theorem 3. *Given m/n, with $1 \leqslant m \leqslant n$ and $\gcd(m, n) = 1$. Consider the equation*

$$1/x + 1/y = m/n.$$

If (x, y) is a positive solution of the equation then there are positive integers f_1, f_2 such that $f_1 f_2 = n^2$, $m \mid \gcd(n + f_1, n + f_2)$ and $x = (n + f_1)/m$, $y = (n + f_2)/m$.

Conversely, if there exist positive integers f_1, f_2 such that $f_1 f_2 = n^2$, and m divides both $n + f_1$ and $n + f_2$, then $x = (n + f_1)/m$, $y = (n + f_2)/m$ is a positive solution of the equation.

Proof. If (x, y) is a positive solution to $1/x + 1/y = m/n$, then $1/xm + 1/ym = 1/n$. By Theorem 2, there are positive f_1 and f_2 with $f_1 f_2 = n^2$, such that $(xm, ym) = (n + f_1, n + f_2)$. As m divides both $n + f_1$ and $n + f_2$, then $m \mid \gcd(n + f_1, n + f_2)$. Moreover, (x, y) is equal to $((n + f_1)/m, (n + f_2)/m)$.

Conversely, suppose f_1 and f_2 are positive, $f_1 f_2 = n^2$ and $m \mid \gcd(n + f_1, n + f_2)$. By Theorem 2, $1/(n + f_1) + 1/(n + f_2) = 1/n$. Hence

$$\frac{1}{(n + f_1)/m} + \frac{1}{(n + f_2)/m} = \frac{m}{n}.$$

That is, $((n + f_1)/m, (n + f_2)/m)$ is a positive solution to the equation. □

The proof includes a method for finding all two-term expansions of a given m/n, or determining that there are none. The method is based on finding all factorizations $n^2 = f_1 f_2$.

Example 2. Find all two-term expansions for $4/25$.

Expansions are obtained from $x = (25 + f_1)/4$, $y = (25 + f_2)/4$ where $f_1 f_2 = 25^2$.

f_1	f_2	$25 + f_1$	$25 + f_2$	x, y
1	625	26	650	–
5	125	30	150	–
25	25	50	50	–

We see that there are no two-term Egyptian fractions for $4/25$.

Example 3. Find all two-term expansions for $3/10$.

Expansions are obtained from $x = (10 + f_1)/3$, $y = (10 + f_2)/3$ where $f_1 f_2 = 10^2$.

f_1	f_2	$10 + f_1$	$10 + f_2$	x, y
1	100	10	110	–
2	50	12	60	$4, 20$
4	25	14	35	–
5	20	15	30	$5, 10$
10	10	20	20	–

Hence there are two two-term Egyptian fractions for $3/10$, namely

$$\frac{3}{10} = \left[\frac{1}{4} + \frac{1}{20}\right] = \left[\frac{1}{5} + \frac{1}{10}\right].$$

The following result can be found in Rav [4]. The result is a slight restatement of Theorem 3.

Theorem 4. *Given a fraction m/n, with $1 \leqslant m \leqslant n$ and $\gcd(m, n) = 1$. Then (x, y) is a positive solution to the equation $1/x + 1/y = m/n$ if and only if there exist positive integers d_1 and d_2 such that $d_1 \mid n$, $d_2 \mid n$, $\gcd(d_1, d_2) = 1$, $m \mid (d_1 + d_2)$, and $x = n(d_1 + d_2)/d_1 m$, $y = n(d_1 + d_2)/d_2 m$.*

Proof. If (x, y) is a positive solution to the equation $1/x + 1/y = m/n$ then by Theorem 3,

$$(x, y) = ((n + f_1)/m, (n + f_2)/m)$$

for some f_1, f_2 with $f_1 f_2 = n^2$. Hence

$$\frac{1}{n\left(1 + \dfrac{f_1}{n}\right)/m} + \frac{1}{n\left(1 + \dfrac{f_2}{n}\right)/m} = \frac{m}{n}.$$

Reduce f_1/n to the lowest terms d_2/d_1. Since $(f_1/n)(f_2/n) = f_1 f_2/n^2 = 1$, we see that $f_2/n = d_1/d_2$, in lowest terms. In particular, $d_1 \mid n$ and $d_2 \mid n$. Moreover

$$\frac{m}{n} = \frac{1}{\dfrac{n\left(1 + \dfrac{d_2}{d_1}\right)}{m}} + \frac{1}{\dfrac{n\left(1 + \dfrac{d_1}{d_2}\right)}{m}} = \frac{1}{\dfrac{n\,(d_1 + d_2)}{d_1 m}} + \frac{1}{\dfrac{n\,(d_1 + d_2)}{d_2 m}}.$$

Since the denominators in the fractions on the right hand side of the above equation are positive integers, and $\gcd(m,n) = 1$, we conclude that $m \mid (d_1 + d_2)$.

The converse is easy. □

According to Theorem 4, to find the solutions to the equation $1/x + 1/y = m/n$, we just need to find all the ordered pairs of coprime factors (d_1, d_2) of n, and then for each pair check whether $m \mid (d_1 + d_2)$. Note that $(1, 1)$ is a pair of coprime factors of n. Whenever $m \mid (d_1 + d_2)$, then $((n/d_1)((d_1 + d_2)/m), (n/d_2)((d_1 + d_2)/m))$ is a solution to the equation.

Example 4. Reconsider the equation $1/x + 1/y = 4/25$ given in Example 2. The positive factors of 25 are 1, 5, and 25. The ordered pairs of coprime factors are $(d_1, d_2) = (1, 1)$, (1, 5), (5, 1), (1, 25), and (25, 1). The values of the sums are $d_1 + d_2 =$ 2, 6, and 26, none of which are divisible by 4. Therefore the equation does not have positive solutions.

Example 5. Find all two-term expansions for $3/10$.

These are given from $x = \frac{10}{d_1}\frac{d_1+d_2}{3}$, $y = \frac{10}{d_2}\frac{d_1+d_2}{3}$ where $d_1 \,|10$, $d_2 \,|10$, $(d_1, d_2) = 1$, and $3 \mid (d_1 + d_2)$.

d_1	d_2	$d_1 + d_2$	x, y
1	1	1	–
1	2	3	$10, 5$
1	5	6	$20, 4$
1	10	11	–
2	5	7	–

The results agree with Example 3.

Considering that $m = 1$ divides every sum of coprime factors of n, we get a relation between the number of positive factors of n^2 and the number of pairs of coprime factors of n.

Theorem 5. *The number of positive factors of n^2 is equal to the number of ordered pairs of positive coprime factors of n.*

3 Proper two-term Egyptian fractions

In order to count the number of two-term Egyptian fractions, we will not distinguish $1/a + 1/b$ from $1/b + 1/a$. Identical unit fractions are also excluded from consideration. In view of Theorem 2, we have

Theorem 6. *The number of proper two-term Egyptian fractions for the unit fraction $1/n$ is $\frac{1}{2}(\tau(n^2) - 1)$.*

Now consider two-term Egyptian fractions for $2/n$ where $\gcd(2, n) = 1$. When n is odd, all the factors of n are odd, therefore in view of Theorems 3 and 4 we obtain:

Theorem 7. *There are $\frac{1}{2}(\tau(n^2) - 1)$ proper two-term Egyptian fractions for the fraction $2/n$, where $n > 2$ and $\gcd(2, n) = 1$.*

In addition, all proper two-term Egyptian fractions can be obtained by either one of the following formulae:

(i)

$$\frac{2}{n} = \frac{1}{\frac{1}{2}(n+f_1)} + \frac{1}{\frac{1}{2}(n+f_2)}$$

where $f_1, f_2 > 0$, $f_1 f_2 = n^2$ *and* $f_1 < f_2$*,*

(ii)

$$\frac{2}{n} = \frac{1}{\left(\frac{n}{d_1}\frac{d_1+d_2}{2}\right)} + \frac{1}{\left(\frac{n}{d_2}\frac{d_1+d_2}{2}\right)}$$

where $d_1, d_2 > 0$, $d_1 \,|\, n$, $d_2 \,|\, n$, $d_1 < d_2$, *and* $\gcd(d_1, d_2) = 1$.

Example 6. Find all proper two-term Egyptian fractions for $2/15$.

There are four proper two-term expansions.
Method I: We obtain the expansions from $x = \frac{1}{2}(15+f_1)$, $y = \frac{1}{2}(15+f_2)$ where $f_1 f_2 = 3^2 \cdot 5^2$.

f_1	f_2	x	y
1	225	8	120
3	75	9	45
5	45	10	30
9	25	12	20

Method II: We obtain the expansions from $x = \dfrac{n}{d_1}\dfrac{d_1+d_2}{2}$, $y = \dfrac{n}{d_2}\dfrac{d_1+d_2}{2}$, where $d_1 \mid 15$, $d_2 \mid 15$, and $(d_1, d_2) = 1$.

d_1	d_2	x	y
1	3	30	10
1	5	45	9
3	5	20	12
1	15	120	8

Both methods result in

$$\frac{2}{15} = \left[\frac{1}{9} + \frac{1}{45}\right] = \left[\frac{1}{8} + \frac{1}{120}\right] = \left[\frac{1}{10} + \frac{1}{30}\right] = \left[\frac{1}{12} + \frac{1}{20}\right].$$

The next is a simple consequence of Theorem 7.

Theorem 8. *Every rational* $2/n$*, where* $2 < n$ *and* $\gcd(2, n) = 1$*, admits a proper two-term Egyptian fraction.*

Moreover, $2/n$ *admits a unique proper two-term Egyptian fraction if and only if* $n = p$ *is an odd prime. Namely*

$$\frac{2}{p} = \frac{2}{p+1} + \frac{2}{p(p+1)}.$$

Remark 1. In the Rhind Mathematical Papyrus Table [2], the Egyptians gave an Egyptian fraction for each of $2/n$, where n is odd and $n \leqslant 101$. When n is a multiple of 3 or 5, except $n = 95$, a two-term Egyptian fraction was given. When n is a prime number, except $n = 3$, 5, 7, 11 and

23, the Egyptian fraction stated had either three or four terms. We do not know whether ancient Egyptians pursued two-term expressions for $2/n$. However, we are able to provide two-term expressions for those cases mentioned above where two-term expressions were not in the table, for completeness.

When $n = 3$, 5, 7, 11 and 23, there is only one proper two-term expression for $2/n$, namely, $2/n = 2/(n+1) + 2/(n+n^2)$.

When $n = 95$, there are $(\tau(95^2) - 1)/2 = 4$ proper two-term Egyptian fractions for $2/n = 2/95$. With either method introduced in Theorem 7, we get $2/95 = [1/48 + 1/4560] = [1/50 + 1/950] = [1/57 + 1/285] = [1/60 + 1/228]$.

When $m \neq 2$ then m/n may not admit two-term expansions. Necessary and sufficient conditions for existence of two-term expansions will be given in the special case when the prime factors of n are congruent to $\pm 1 \pmod{m}$.

Theorem 9. *Let* $2 < m < n$, $\gcd(m, n) = 1$. *Suppose that each prime factor of* n *is congruent to either* 1 *or* $-1 \pmod{m}$. *Then* m/n *has a proper two-term Egyptian fraction if and only if* n *has a prime factor congruent to* $-1 \pmod{m}$.

Proof. By hypothesis, if p is a prime divisor of n then $p \equiv \pm 1 \pmod{m}$. Hence

$$n \equiv \pm 1 \pmod{m}.$$

Consider the case that n has a prime factor $d_1 \equiv -1 \pmod{m}$. Let $d_2 = 1$. Then $\gcd(d_1, d_2) = 1$ and $m \mid (d_1 + d_2)$. By Theorem 4, m/n has a two-term Egyptian fraction, and it is proper since $d_1 \neq d_2$.

Conversely, if m/n has a two-term Egyptian fraction, then by Theorem 3, the two denominators are equal to $(n + f_1)/m$ and $(n + f_2)/m$ respectively, where $f_1 f_2 = n^2$ and both $n + f_1$ and $n + f_2$ are multiples of m, whence $f_1 \equiv -n \pmod{m}$. If $n \equiv -1 \pmod{m}$ then n has a prime factor congruent to $-1 \pmod{m}$. On the other hand if $n \equiv 1 \pmod{m}$ then $f_1 \equiv -1 \pmod{m}$. But the factors of f_1 are factors of n, therefore f_1, and hence n also, has a prime factor congruent to $-1 \pmod{m}$. □

When $m = 3$, 4 or 6, and $2 < m < n$ with $\gcd(m, n) = 1$, then any prime divisor of n is necessarily congruent to $\pm 1 \pmod{m}$. Hence we obtain results for proper fractions $3/n$, $4/n$, $6/n$ in lowest terms:

Theorem 10. (i) *The fraction* $3/n$ *has a proper two-term Egyptian fraction if and only if* n *has a prime factor congruent to* $2 \pmod{3}$.

(ii) *The fraction* $4/n$ *has a proper two-term Egyptian fraction if and only if* n *has a prime factor congruent to* $3 \pmod{4}$.

(iii) *The fraction* $6/n$ *has a proper two-term Egyptian fraction if and only if* n *has a prime factor congruent to* $5 \pmod{6}$.

Remark 2. (i) When all the prime factors of n are congruent to $1 \pmod{3}$ then it is known that $3/n$ has a three term expansion.

(ii) When all the prime factors of n are congruent to $1 \pmod 4$ then it is conjectured that $4/n$ has a three term expansion [Erdős–Straus].

(iii) When all the prime factors of n are congruent to $1 \pmod 6$ then it is open what is a minimal-length expansion of $6/n$.

We next obtain formulae for the number of proper two-term Egyptian fractions for m/n, and also a method for obtaining these expansions.

Notation. We define $\kappa(1) = 0$ and if $d = \prod_{j=1}^{k} q_j^{r_j}$, a prime-power factorization, then $\kappa(d) = \sum_{j=1}^{k} r_j$. We define $O(n) = |\{d : d \,|\, n, \kappa(d) \text{ is odd}\}|$, which is the number of factors of n containing an odd number of prime factors, and $E(n) = |\{d : d \,|\, n, \kappa(d) \text{ is even}\}|$, which is the number of factors of n containing an even number of prime factors.

Lemma 11. *Let $n = \prod_{j=1}^{k} q_j^{2s_j}$ where $k \geqslant 1$, $s_j \geqslant 1$, and q_j are distinct primes. Then*

(i) $O(n) = (\tau(n) - 1)/2$.

(ii) $E(n) = (\tau(n) + 1)/2$.

Proof. We prove (i) and (ii) simultaneously by induction on k. First, $\tau(q^{2s}) = 2s + 1$. The factors of q^{2s} are $q^0, q^1, \ldots, q^{2s}$. It is easy to see that formulae are true when $k = 1$.

Now suppose formulae are true for a fixed $k \geqslant 1$, and let $n = \prod_{j=1}^{k+1} q_j^{2s_j} = a \cdot b$ where $a = \prod_{j=1}^{k} q_j^{2s_j}$, $b = q_{k+1}^{2s_{k+1}}$. The formulae for $O(a)$, $E(a)$ are given by induction hypothesis, and for $O(b)$, $E(b)$ given by the case $k = 1$. Since $\gcd(a, b) = 1$, then $\tau(ab) = \tau(a)\tau(b)$. We have

$$\begin{aligned} O(n) &= O(a)E(b) + E(a)O(b) \\ &= \frac{(\tau(a) - 1)}{2}\frac{(\tau(b) + 1)}{2} + \frac{(\tau(a) + 1)}{2}\frac{(\tau(b) - 1)}{2} \\ &= \frac{\tau(a)\tau(b) - 1}{2} = \frac{\tau(ab) - 1}{2} = \frac{\tau(n) - 1}{2}. \end{aligned}$$

Then $E(n) = \tau(n) - O(n) = (\tau(n) + 1)/2$. □

Theorem 12. *Suppose m/n, $2 < m < n$ and $\gcd(m, n) = 1$, is a fraction such that all the prime factors of n are congruent to either 1 or $-1 \pmod m$. Let p be the product of the prime factors of n congruent to 1, and q the product of the prime factors congruent to -1. The empty product is defined to be 1.*

(i) *If $n \equiv 1 \pmod m$, then the number of proper two-term Egyptian fractions of m/n is equal to $\tau(p^2)(\tau(q^2) - 1)/4$.*

When $q = 1$ then this number is zero. If $q \neq 1$, to get a two-term Egyptian fraction, factor n^2 as prime powers and write $n^2 = f_1 f_2$, such that f_1 and f_2 both contain an odd number of n^2's prime factors that are congruent to $-1 \pmod m$.

(ii) *If $n \equiv -1 \pmod m$, then the number of proper two-term Egyptian fractions of m/n is equal to $\tau(p^2)(\tau(q^2) + 1)/4$.*

To get a two-term Egyptian fraction, factor n^2 as prime powers and write $n^2 = f_1 f_2$, such that f_1 and f_2 both contain an even number of n^2's factors that are congruent to $-1 \pmod{m}$.

In both cases,

$$\frac{m}{n} = \frac{1}{(n+f_1)/m} + \frac{1}{(n+f_2)/m}$$

is a proper two-term Egyptian fraction.

Proof. By hypothesis we may write

$$n = \prod_{i=1}^{k} p_i^{r_i} \prod_{j=1}^{l} q_j^{s_j} = p \cdot q$$

where $k, l \geqslant 0$, $r_i \geqslant 1$, $s_j \geqslant 1$, p_i, q_j prime, $p_i \equiv 1$, $q_j \equiv -1 \pmod{m}$, and the empty product is defined to be 1. Put $p = \prod_{i=1}^{k} p_i^{r_i}$, $q = \prod_{j=1}^{l} q_j^{s_j}$.

Define $\sigma = 0$ if $l = 0$, and $\sigma = s_1 + \cdots + s_l$ if $l > 0$. We see that $n \equiv (-1)^\sigma \pmod{m}$. By Theorem 9, m/n has a proper two term expansion if and only if $\sigma > 0$.

Case (i): Suppose $n \equiv 1 \pmod{m}$. Then σ is even. If $\sigma = 0$, then m/n does not have a proper two-term Egyptian fraction. The number $\tau(p^2)(\tau(q^2)-1)/4 = \tau(p^2)(\tau(1)-1)/4 = 0$, and the statement is true. Now suppose $\sigma \geqslant 2$ (and even), so that m/n has a proper two-term Egyptian fraction, and also $q \neq 1$. Consider the equation $1/x + 1/y = m/n$.
By Theorem 3, the positive integer solutions are given by $((n+f_1)/m, (n+f_2)/m)$, where $f_1, f_2 > 0$, $f_1 f_2 = n^2 = \prod_{i=1}^{k} p_i^{2r_i} \prod_{j=1}^{l} q_j^{2s_j}$ and $n + f_i \equiv 0 \pmod{m}$ $(i = 1, 2)$, with $f_i \equiv -1$, since $n \equiv 1$. Hence we may write $f_1 = \prod_i p_i^{\alpha_i} \prod_j q_j^{\beta_j}$ where $0 \leqslant \alpha_i \leqslant 2r_i$ and $0 \leqslant \beta_j \leqslant 2s_j$ (all i, all j), and $\sum_j \beta_j$ odd. Since $f_1 f_2 = n^2$, we deduce that $f_2 = \prod_i p_i^{2r_i - \alpha_i} \prod_j q_j^{2s_j - \beta_j}$, where $\sum_j (2s_j - \beta_j)$ is odd. Now q^2 has $\tau(q^2) = (2s_1+1)\cdots(2s_l+1)$ factors. Among these factors of q^2, using Lemma 11, there are $(\tau(q^2)-1)/2$ factors containing an odd number of prime factors q_j. The number $(\tau(q^2)-1)/2$ is non-zero. Thus the number of possible f_1 is $\tau(p^2)(\tau(q^2)-1)/2$. If (f_1, f_2) generates a solution then also does (f_2, f_1). Notice that f_1 and f_2 can never be equal, since otherwise $f_1 = f_2 = pq \equiv 1 \pmod{m}$, a contradiction. Omitting duplications, we get that m/n has $\tau(p^2)(\tau(q^2)-1)/4$ proper two-term Egyptian fractions.

Case (ii): Suppose $n \equiv -1 \pmod{m}$. Then σ is odd. By Theorem 9 the fraction m/n has a proper two-term Egyptian fraction. Consider the equation $1/x + 1/y = m/n$.
By Theorem 3, every positive integer solution can be expressed as $\big((n+f_1)/m, (n+f_2)/m\big)$, where $f_1 f_2 = n^2 = \prod_{i=1}^{k} p_i^{2r_i} \prod_{j=1}^{l} q_j^{2s_j}$, and both f_1, $f_2 \equiv -n \equiv 1 \pmod{m}$. Hence we may write $f_1 = \prod_i p_i^{\alpha_i} \prod_j q_j^{\beta_j}$ where $0 \leqslant \alpha_i \leqslant 2r_i$ and $0 \leqslant \beta_j \leqslant 2s_j$, and $\sum \beta_j$ even. Given f_1, then $f_2 = \prod_i p_i^{2r_i - \alpha_i} \prod_j q_j^{2s_j - \beta_j}$, and $\sum_j (2s_j - \beta_j)$ is even. Among the factors of q^2, using Lemma 11, there are $(\tau(q^2)+1)/2$ factors containing an even number of prime factors. The number $(\tau(q^2)+1)/2$ is non-zero. Hence f_1 has $\tau(p^2)(\tau(q^2)+1)/2$ candidates. If (f_1, f_2) gives a solution then also does (f_2, f_1). Notice that f_1 and f_2 can never be equal, since otherwise $f_1 = f_2 = pq \equiv -1 \pmod{m}$, a contradiction. Omitting duplications, we get that m/n has $\tau(p^2)(\tau(q^2)+1)/4$ proper two-term Egyptian fractions. □

Notice that, given m/n with $2 < m < n$ and $\gcd(m, n) = 1$, the hypothesis of Theorem 12 that each prime factor of n is congruent to $\pm 1 \pmod{m}$ is satisfied when $m = 3$, 4, or 6.

Example 7. Find all proper two-term expansions for $3/245$.

Apply Theorem 12. Note $n = 5 \cdot 7^2$. Since $q = 5$, $p = 49$, and $245 \equiv -1 \pmod{3}$, we use Theorem 12 (ii). The number of proper two-term expansions is $\tau(7^4)(\tau(5^2) + 1)/4 = 5$. Moreover they can be obtained from $x = (245 + f_1)/3$, $y = (245 + f_2)/3$ where $f_1 = 5^0 k$, $f_2 = 5^2 l$ and $kl = 7^4$.

f_1	f_2	x	y
1	$5^2.7^4$	82	20090
7	$5^2.7^3$	84	2940
7^2	$5^2.7^2$	98	490
7^3	$5^2.7$	196	140
7^4	5^2	882	90

Hence

$$\frac{3}{245} = \left[\frac{1}{82} + \frac{1}{20090}\right] = \left[\frac{1}{84} + \frac{1}{2940}\right] = \left[\frac{1}{98} + \frac{1}{490}\right] = \left[\frac{1}{140} + \frac{1}{196}\right] = \left[\frac{1}{90} + \frac{1}{882}\right].$$

Example 8. Find all proper two-term expansions for $4/245$.

We find $n = 5 \cdot 7^2$, $p = 5$, $q = 7^2$, $n \equiv 1 \pmod{4}$. By Theorem 12 (i), the number of proper two-term expansions is $\tau(5^2)(\tau(7^4) - 1)/4 = 3$. There are obtained from $x = (245 + f_1)/4$, $y = (245 + f_2)/4$ where $f_1 = 7^1 k$, $f_2 = 7^3 l$ and $kl = 5^2$.

f_1	f_2	x	y
$7 \,.\, 1$	$7^3 \,.\, 5^2$	63	2205
$7 \,.\, 5$	$7^3 \,.\, 5$	70	490
$7 \,.\, 5^2$	$7^3 \,.\, 1$	105	147

Hence

$$\frac{4}{245} = \left[\frac{1}{63} + \frac{1}{2205}\right] = \left[\frac{1}{70} + \frac{1}{490}\right] = \left[\frac{1}{105} + \frac{1}{147}\right].$$

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