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The structure of the Fibonacci numbers in the modular ring Z_5

J. V. Leyendekkers[1] and A. G. Shannon[2]

[1] Faculty of Science, The University of Sydney
Sydney, NSW 2006, Australia

[2] Faculty of Engineering & IT, University of Technology
Sydney, NSW 2007, Australia

e-mails: `tshannon38@gmail.com`, `anthony.shannon@uts.edu.au`

Abstract: Various Fibonacci number identities are analyzed in terms of their underlying integer structure in the modular ring Z_5.

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1 Introduction

Over eight centuries ago in Chapter XII of his book, *Liber Abaci*, Leonardo of Pisa (nicknamed Fibonacci) presented and solved his famous problem on the reproduction of rabbits in terms of the famous sequence which bears his name. Four centuries later, Albert Girard in 1634 gave the notation for the recurrence relation for the terms of the sequence in use today, namely

$$F_{n+1} = F_n + F_{n-1}. \tag{1.1}$$

Over the centuries since the Fibonacci sequence of integers has been applied to a myriad of mathematical applications, especially in number theory [1]. In particular, Kepler [6] observed that the ratio of consecutive Fibonacci numbers converges to the Golden Ratio φ. He also showed that the square of any term differs by unity from the product of the two adjacent terms in the sequence (Simson's or Cassini's Identity (3.2) below).

In this paper we analyse the structure of the Fibonacci sequence in the context of the modular ring Z_5 (Table 1) [2]. The underlying structure accounts for many of the unique properties of this fascinating sequence, particularly their congruence properties [7].

Class	$\bar{0}_5$	$\bar{1}_5$	$\bar{2}_5$	$\bar{3}_5$	$\bar{4}_5$
Row	**$5r_0$**	**$5r_1$+1**	**$5r_2$+2**	**$5r_3$+3**	**$5r_4$+4**
0	0	1	2	3	4
1	5	6	7	8	9
2	10	11	12	13	14
3	15	16	17	18	19
4	20	21	22	23	24

Table 1. Rows of modular ring Z_5

2 Class patterns of Fibonacci numbers

The pattern of the Fibonacci numbers in Z_5 is displayed in Table 2.

n	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Z_5	$\bar{1}_5$	$\bar{1}_5$	$\bar{2}_5$	$\bar{3}_5$	$\bar{0}_5$	$\bar{3}_5$	$\bar{3}_5$	$\bar{1}_5$	$\bar{4}_5$
10	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**
$\bar{0}_5$	$\bar{4}_5$	$\bar{4}_5$	$\bar{3}_5$	$\bar{2}_5$	$\bar{0}_5$	$\bar{2}_5$	$\bar{2}_5$	$\bar{4}_5$	$\bar{1}_5$
n	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**
Z_5	$\bar{0}_5$	$\bar{1}_5$	$\bar{1}_5$	$\bar{2}_5$	$\bar{3}_5$	$\bar{0}_5$	$\bar{3}_5$	$\bar{3}_5$	$\bar{1}_5$
29	**30**	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**
$\bar{4}_5$	$\bar{0}_5$	$\bar{4}_5$	$\bar{4}_5$	$\bar{3}_5$	$\bar{2}_5$	$\bar{0}_5$	$\bar{2}_5$	$\bar{2}_5$	$\bar{4}_5$

Table 2. Fibonacci numbers in Z_5

The patterns of the modular residues follow the form $\overline{N}_5\bar{0}_5\overline{N}_5\overline{N}_5\overline{M}_5$ in which the numbers $\overline{N}_5$ have the pattern $\bar{1}_5\bar{3}_5\bar{4}_5\bar{2}_5$ and the interstitial numbers $\overline{M}_5$ have the pattern $\bar{2}_5\bar{1}_5\bar{3}_5\bar{4}_5$. These patterns allow prediction of the class of F_n, and hence the right-end-digit (RED) from n (Table 3).

F_n^*	***n* for $\overline{N}_5$**	***n* for $\overline{M}_5$**
(1,6) $\bar{1}_5$	19, 39, 59, 79, ... 1, 21, 41, 61, ... 2, 22, 42, 62, ...	8, 28, 48, 68, ...
(2,7) $\bar{2}_5$	14, 34, 54, 74, ... 16, 36, 56, 76, ... 17, 37, 57, 77, ...	3, 23, 43, 63, ...
(3,8) $\bar{3}_5$	4, 24, 44, 64, ... 6, 26, 46, 66, ... 7, 27, 47, 67, ...	13, 33, 53, 73, ...
(4,9) $\bar{4}_5$	9, 29, 49, 69, ... 11, 31, 51, 71, ... 12, 32, 52, 72, ...	18, 38, 58, 78, ...
(0,5) $\bar{0}_5$	0,5,10,15,20,...	

Table 3. Details of the patterns (F_n^*: Class of F_n)

There are many characteristics of the Fibonacci sequence that are directly related to this structure. We consider some of them here. They serve as examples for further analysis. Another approach would be to consider the algebra of $F(\sqrt{5})$ where $F(x)$ is the characteristic polynomial associated with the recurrence relation (1.1) and is irreducible in the field F of its coefficients [8].

3 The relationship to the Golden Ratio

If the measure of a line AB is given by F_{n+1},nd AB is divided into two different sized segments, AC and CB, with CB>AC, then AB/CB = CB/AC approximately defines φ, the Golden Ratio if CB = F_n and AC = F_{n-1}, so that approximately

$$F_{n+1}F_{n-1} \cong F_n^2 \tag{3.1}$$

However, as first noted by Kepler the two sides of (3.1) always differ by unity as we can see from the class structures (Table 4).

n	F_n^*	$\left(F_n^2\right)^*$	F_{n+1}^*	F_{n-1}^*	$\left(F_{n+1}F_{n-1}\right)^*$
4	3,8	9,4	0,5	2,7	0,5
41	1,6	1,6	1,6	0,5	0,5
77	2,7	4,9	4,9	2,7	3,8
92	4,9	6,1	3,8	4,9	7,2

Table 4. Data from Tables 2, 3

The equality is expressed in Simson's Identity

$$F_{n+1}F_{n-1} = F_n^2 + (-1)^n \tag{3.2}$$

$$F_{n+1}/F_n = F_n / F_{n-1} + (-1)^n / \left(F_n F_{n-1}\right) \tag{3.3}$$

of which the second term on the right hand side is very small for large n; this is the error term in the Fibonacci approximation for φ [5].

$$F_{n+6} = 4F_{n+3} + F_n \tag{3.4}$$

so that

$$F_{n+6}/F_n - 4F_{n+3}/F_n = 1, \tag{3.5}$$

which when substituted into (3.2) yield

$$F_{n+1}F_nF_{n-1} = \begin{cases} F_n^3 + F_{n+6} - 4F_{n+3}, & n \textit{ even}, \\ F_n^3 - F_{n+6} + 4F_{n+3,} & n \textit{ odd}. \end{cases} \tag{3.6}$$

Many other elegant relationships can be formed in this way.

4 “Squaring” rectangles

An odd number of golden rectangles with sides equal to successive Fibonacci numbers can appear to fit into squares as “demonstrated” in Figure 1.

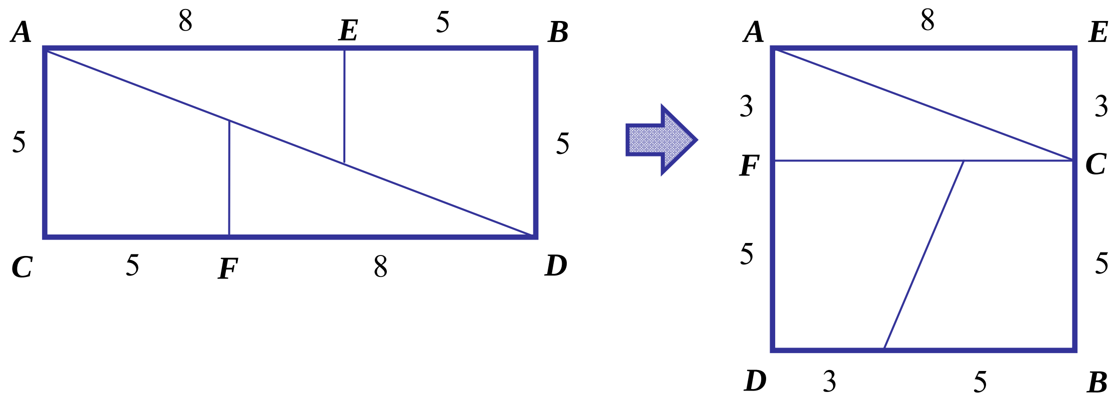

Figure 1. “Squaring” the Golden Rectangle

This is not drawn to scale, but essentially a golden rectangle of sides F_5 and F_7 units, and hence of area 65 square units, is transformed into a square of side F_6 units and hence of area 64 square units. Of course, while the eye might just be deceived, Simson is not! From Simson’s identity we get for odd *n* that

$$F_{n-1}^2 + F_n F_{n-1} + F_n F_{n+1} = F_{n+1}^2 \tag{4.1}$$

The structure of the Fibonacci sequence in Z_5 in Section 2 shows that this square sum depends on the constraints of the squares which occur only in Classes $\overline{0}_5$, $\overline{1}_5$ and $\overline{4}_5$, and the sums are also confined to these classes in harmony with this square (Table 5).

Number of Products *n*	Classes of F_{n+1}	Classes of F_{n+1}^2	Class of $F_{n-1}^2 + F_n F_{n-1} + F_n F_{n+1}$
3	$\overline{3}_5$	$\overline{4}_5$	$\overline{1}_5+\overline{2}_5+\overline{1}_5=\overline{4}_5$
5	$\overline{3}_5$	$\overline{4}_5$	$\overline{4}_5+\overline{0}_5+\overline{0}_5=\overline{4}_5$
7	$\overline{1}_5$	$\overline{1}_5$	$\overline{4}_5+\overline{4}_5+\overline{3}_5=\overline{1}_5$
9	$\overline{0}_5$	$\overline{0}_5$	$\overline{1}_5+\overline{4}_5+\overline{0}_5=\overline{0}_5$
11	$\overline{4}_5$	$\overline{1}_5$	$\overline{0}_5+\overline{0}_5+\overline{1}_5=\overline{1}_5$
13	$\overline{2}_5$	$\overline{4}_5$	$\overline{1}_5+\overline{2}_5+\overline{1}_5=\overline{4}_5$
15	$\overline{2}_5$	$\overline{4}_5$	$\overline{4}_5+\overline{0}_5+\overline{0}_5=\overline{4}_5$
17	$\overline{4}_5$	$\overline{1}_5$	$\overline{4}_5+\overline{4}_5+\overline{3}_5=\overline{1}_5$
19	$\overline{0}_5$	$\overline{0}_5$	$\overline{1}_5+\overline{4}_5+\overline{0}_5=\overline{0}_5$
21	$\overline{1}_5$	$\overline{1}_5$	$\overline{0}_5+\overline{0}_5+\overline{1}_5=\overline{1}_5$

Table 5. Classes of Sums from (4.1)

5 The Factor 11

The result

$$\lim_{n\to\infty} F_{n+6} / F_n = 8\varphi + 5$$

from [4] was obtained from the above characteristics of the sequence. Moreover, the Class of the sum of ten consecutive integers is the same as the class of the seventh number in the ten. The seventh number times 11 equals the sum of the ten. This is consistent with $11 \in \bar{1}_5$ and $\bar{1}_5 \times \bar{a}_5 = \bar{a}_5$ (Table 6). Note that the RED of the sum is the same as the RED of the seventh number, and since the RED of 11 is 1 it is the only integer to satisfy.

Range of n	Class of sum	Class of 7th Integer, N_7	Class of $\bar{1}_5 \times N_7$
1 – 10	$\bar{3}_5$	$\bar{3}_5$	$\bar{3}_5$
2 – 11	$\bar{1}_5$	$\bar{1}_5$	$\bar{1}_5$
3 – 12	$\bar{4}_5$	$\bar{4}_5$	$\bar{4}_5$
4 – 13	$\bar{0}_5$	$\bar{0}_5$	$\bar{0}_5$
5 – 14	$\bar{4}_5$	$\bar{4}_5$	$\bar{4}_5$
6 – 15	$\bar{4}_5$	$\bar{4}_5$	$\bar{4}_5$
7 – 16	$\bar{3}_5$	$\bar{3}_5$	$\bar{3}_5$
8 – 17	$\bar{2}_5$	$\bar{2}_5$	$\bar{2}_5$
9 – 18	$\bar{0}_5$	$\bar{0}_5$	$\bar{0}_5$
10 – 19	$\bar{2}_5$	$\bar{2}_5$	$\bar{2}_5$

Table 6. Class structure in sets of 10 integers

The class structure of the 7th number in each set of ten integers is:

$$\bar{3}_5 \quad \bar{1}_5 \quad \bar{4}_5 \quad \bar{0}_5 \quad \bar{4}_5 \quad \bar{4}_5 \quad \bar{3}_5 \quad \bar{2}_5 \quad \bar{0}_5 \quad \bar{2}_5$$

which corresponds to F_7 on the F_n class pattern in Section 2.

6 Periodicity of Fibonacci number right end digits

A RED periodicity of 60 for integers was discovered in general in 1774 by Joseph Louis Lagrange [6]. However, this periodicity pattern is more complicated than previously assumed for the Fibonacci sequence. For even REDs the interval is 60, but for odd REDs the intervals can be 20 or 40 which indeed sum to 60, and for Class $\bar{0}_5$ the intervals are 30 (Table 7).

Class of F_n	F_n^*	n^*	n	Δn
$\overline{0}_5$	0	0	30, 60, 90, 120, 150	30, 30, 30, 30
	5	5	15, 45, 75, 105, 135	30, 30, 30, 30
$\overline{1}_5$	6	1	21, 81, 141, 201	60, 60, 60, 60
		2	42, 102, 162, 222	60, 60, 60, 60
		8	48, 108, 168, 228	60, 60, 60, 60
		9	39, 99, 159, 219	60, 60, 60, 60
	1	1	1, 41, 61, 101, 121	40, 20, 40, 20
		2	2, 22, 62, 82, 122	20, 40, 20, 40
		8	8, 28, 68, 88, 128	20, 40, 20, 40
		9	19, 59, 79, 119, 139	40, 20, 40, 20
$\overline{2}_5$	2	3	3, 63, 123, 183	60, 60, 60, 60
		4	54, 114, 174, 234	60, 60, 60, 60
		6	36, 96, 156, 216	60, 60, 60, 60
		7	57, 117, 177, 237	60, 60, 60, 60
	7	3	23, 43, 83, 103, 143	20, 40, 20, 40
		4	14, 34, 74, 94, 134	20, 40, 20, 40
		6	16, 56, 76, 116, 136	40, 20, 40, 20
		7	17, 37, 77, 97, 137	20, 40, 20, 40
$\overline{3}_5$	8	3	33, 93, 153, 213	60, 60, 60, 60
		4	24, 84, 144, 204	60, 60, 60, 60
		6	6, 66, 126, 186	60, 60, 60, 60
		7	27, 87.147, 207	60, 60, 60, 60
	3	3	13, 53, 73, 113, 133	40, 20, 40, 20
		4	4, 44, 64, 104, 124	40, 20, 40, 20
		6	26, 46, 86, 106, 146	20, 40, 20, 40
		7	7, 47, 67, 107, 127	40, 20, 40, 20
$\overline{4}_5$	4	1	51, 111, 171, 231	60, 60, 60, 60
		2	12, 72, 132, 192	60, 60, 60, 60
		8	18, 78, 138, 198	60, 60, 60, 60
		9	9, 69, 129, 189	60, 60, 60, 60
	9	1	11, 31, 71, 91, 131	20, 40, 20, 40
		2	32, 52, 92, 112, 152	20, 40, 20, 40
		8	38, 58, 98, 118, 158	20, 40, 20, 40
		9	29, 49, 89, 109, 149	20, 40, 20, 40

Table 7. Periodicities of Fibonacci REDs

7 Final comments

The structure of the F_n^2 sequence is

$$\overline{1}_5\,\overline{1}_5\,\overline{4}_5\,\overline{4}_5\,\overline{0}_5\,\overline{4}_5\,\overline{4}_5\,\overline{1}_5\,\overline{1}_5\,\overline{0}_5\,\overline{1}_5\,\overline{1}_5\,\overline{4}_5\,\overline{4}_5\,\overline{0}_5\,\overline{4}_5\,\overline{4}_5$$

which follows from the restricted distribution of the squares in Z_5. This simple structure facilitates the formation of Pythagorean triples from F_n [1], and known results such as

$$F_{2n-1} = F_n^2 + F_{n-1}^2, n > 1,$$

and

$$F_{n+2} = 1 + \sum_{j=1}^{n} F_j \ .$$

These can also be related to the structure.

Finally, the interested reader might like to apply the foregoing to the Pellian sequences to compare the similarities and differences with the Fibonacci sequence [3].

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