Notes on Number Theory and Discrete Mathematics
Vol. 19, 2013, No. 1, 59–65

A note on Bernoulli numbers

Ramesh Kumar Muthumalai

Department of Mathematics, D.G. Vaishnav College
Chennai-600106, Tamil Nadu, India
e-mail: ramjan_80@yahoo.com

Abstract: Some explicit formulae for Bernoulli numbers and Bernoulli polynomials are derived. Some rational approimations to powers of π are given in terms of Bernoulli numbers.
Keywords: Bernoulli numbers, Bernoulli polynomials, Stirling numbers, Identities, π.
AMS Classification: 11B68.

1 Introduction

The Bernoulli numbers $B_n, n \in W$ are a sequence of rational numbers with many interesting arithmetic properties. They are defined by the following generating function [5, p. 525].

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_n \frac{t^n}{n!}, \qquad |t| < 2\pi. \tag{1.1}$$

The appearances of Bernoulli numbers throughout mathematics are abundant and include finding a formula for the sum of powers of the first n positive integers, values of L-functions and Euler-Macluarin summation formulae [3]. There exists many recursion and explicit formulae for Bernoulli numbers. The explicit formula in terms of Stirling numbers of second kind given in [5, p. 461] is as follows

$$B_k = \sum_{m=1}^{k} (-1)^m \frac{m!}{m+1} S_m^{(k)}. \tag{1.2}$$

Where $S_m^{(k)}$ striling number of second kind. Also, double series representaion given in [5, p. 537] is as follows

$$B_m = \sum_{k=0}^{m} \sum_{v=0}^{k} (-1)^v \binom{k}{v} \frac{v^m}{k+1}. \tag{1.3}$$

In the present study, similar identities for Bernoulli numbers as shown in (1.2) and (1.3) are derived through Stirling numbers of second kind and double series in different form. Further, some rational approximations to powers of π are given in terms of Bernoulli numbers.

2 Some lemmas and remarks

Lemma 2.1. *Let $n \in W$ and B_n denotes n^{th} Bernoulli number. If*

$$T(x) = \left(\frac{x}{1-e^x}\right)\left[\left(\frac{1-e^x}{x}+1\right)^{n+1}-1\right], \tag{2.1}$$

then

$$T^{(n)}(0) = B_n. \tag{2.2}$$

Where $T^{(n)}(x) = \frac{d^nT(x)}{dx^n}$.

Proof. It is well known that

$$\frac{1-e^x}{x}+1 = -\sum_{k=2}^{\infty}\frac{x^{k-1}}{k!}. \tag{2.3}$$

Differentiating (2.1) n times and using Leibniz rule, then

$$D^nT(x) = \sum_{m=0}^{n}\binom{n}{m}D^m\left(\frac{x}{1-e^x}\right)D^{n-m}\left[\left(\frac{1-e^x}{x}+1\right)^{n+1}-1\right]. \tag{2.4}$$

Where $D^n = \frac{d^n}{dx^n}$. Using (1.1) and (2.3) in (2.4), then setting $x = 0$ and after simplification, completes the Lemma. □

Lemma 2.2. *Let $n \in W$ and $B_n(t)$ denotes n^{th} degree Bernoulli polynomial. If*

$$U(t,x) = e^{tx}T(x), \tag{2.5}$$

then

$$U^{(n)}(t,0) = B_n(t). \tag{2.6}$$

Where $U^{(n)}(x,t) = \frac{d^nU(x,t)}{dx^n}$.

Proof. Given that

$$U(t,x) = e^{tx}T(x).$$

Differentiating n times and using Leibnitz rule, then

$$\frac{d^n}{dx^n}U(t,x)=\sum_{k=0}^{n}\binom{n}{k}t^k e^{tx}\frac{d^{n-k}}{dx^{n-k}}T(x).$$

Putting $x=0$, gives

$$U^{(n)}(t,0)=\sum_{k=0}^{n}\binom{n}{k}t^k T^{(n-k)}(0).$$

Lemma (2.1) shows that $T^{(n-k)}(0)=B_{n-k}$, then gives

$$U^{(n)}(t,0)=\sum_{k=0}^{n}\binom{n}{k}B_{n-k}t^k. \tag{2.7}$$

It is well known that [3, p. 231]

$$B_n(t)=\sum_{k=0}^{n}\binom{n}{k}B_{n-k}t^k. \tag{2.8}$$

Using (2.8) in (2.7) gives (2.6). □

Remark 2.3. Let $m,r\in W$ and $S_{m+r}^{(m)}$ is Stirling number of second kind [2, p. 1037]. Then

$$D^r\left(\frac{e^x-1}{x}\right)^m\Bigg|_{x=0}=\frac{m!r!}{(m+r)!}S_{m+r}^{(m)}. \tag{2.9}$$

Remark 2.4. Let $m,n\in W$ and $B_m^{(-n)}(t)$ is Bernoulli polynomials of degree m of order $-n$ [4]. Then

$$D^r\left(\frac{e^x-1}{x}\right)^m e^{xt}\Bigg|_{x=0}=B_r^{(-n)}(t). \tag{2.10}$$

Where $D^r=\frac{d^r}{dx^r}$.

3 Main Results

Theorem 3.1. *Let $S_m^{(n)}$ be Stirling number of second kind. Then*

$$B_n=n!\sum_{k=0}^{n-1}(-1)^{n-k}\binom{n+1}{k}\frac{(n-k)!}{(2n-k)!}S_{2n-k}^{(n-k)}. \tag{3.1}$$

Proof. From equation (2.1), that

$$T(x)=\left(\frac{x}{1-e^x}\right)\left[\left(\frac{1-e^x}{x}+1\right)^{n+1}-1\right].$$

After simplification, it gives

$$T(x)=\sum_{k=0}^{n}(-1)^{n-k}\binom{n+1}{k}\left(\frac{1-e^{x}}{x}\right)^{n-k}. \tag{3.2}$$

Differentiating (3.2) n times with respect to x and putting $x = 0$ and then using Remark 2.3, yields

$$T^{(n)}(0)=n!\sum_{k=0}^{n}\binom{n+1}{k}\frac{(n-k)!}{(2n-k)!}S_{2n-k}^{(n-k)}.$$

Using Lemma 2.1 and after simplification, gives (3.1). This completes the theorem. □

Example 3.2. Let $n = 1, 2, 3, 4$ in (3.1). Then the first four Bernoulli numbers can be expressed through Stirling numbers of second kind as follows

$$\begin{aligned}
B_1 &= -\frac{1}{2}S_2^{(1)}.\\
B_2 &= \frac{1}{6}S_4^{(2)}-S_3^{(1)}.\\
B_3 &= -\frac{1}{20}S_6^{(3)}+\frac{2}{5}S_5^{(2)}-\frac{3}{2}S_4^{(1)}.\\
B_4 &= \frac{1}{70}S_8^{(4)}-\frac{1}{7}S_7^{(3)}+\frac{2}{3}S_6^{(2)}-2S_5^{(1)}.
\end{aligned}$$

Theorem 3.3. *Let $n \in W$. Then*

$$B_n=n!\sum_{k=0}^{n-1}\sum_{r=0}^{n-k}(-1)^{2n-2k-r}\binom{n+1}{k}\binom{n-k}{r}\frac{r^{2n-k}}{(2n-k)!}. \tag{3.3}$$

Proof. Let us consider the identity given in Theorem 3.1

$$B_n=n!\sum_{k=0}^{n-1}(-1)^{n-k}\binom{n+1}{k}\frac{(n-k)!}{(2n-k)!}S_{2n-k}^{(n-k)}.$$

Using explicit formula for Stirling number of second kind [2, p.1031], then

$$B_n=n!\sum_{k=0}^{n-1}(-1)^{n-k}\binom{n+1}{k}\frac{(n-k)!}{(2n-k)!}\frac{1}{(n-k)!}\sum_{r=0}^{n-k}(-1)^{n-k-r}\binom{n-k}{r}r^{2n-k}.$$

After simplification, it gives

$$B_n=n!\sum_{k=0}^{n-1}\sum_{r=0}^{n-k}(-1)^{2n-2k-r}\binom{n+1}{k}\binom{n-k}{r}\frac{r^{2n-k}}{(2n-k)!}.$$

This completes theorem. □

Theorem 3.4. *Let $B_m^{(-n)}(t)$ be the m^{th} degree Bernoulli polynomial of order $-n$. Then*

$$B_n(t) = \sum_{k=0}^{n} \binom{n+1}{k} B_k^{(k-n)}(t). \tag{3.4}$$

Proof. Let us consider the identity (2.5)

$$U(t,x) = \left(\frac{xe^{xt}}{1-e^x}\right)\left[\left(\frac{1-e^x}{x}+1\right)^{n+1} - 1\right].$$

After simplification, it gives

$$U(t,x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n+1}{k} \left(\frac{1-e^x}{x}\right)^{n-k} e^{xt}.$$

Differentiating above equation n times with respect to x

$$D^n U(t,x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n+1}{k} D^n \left(\frac{1-e^x}{x}\right)^{n-k} e^{xt}.$$

Putting $x = 0$ and then using Remark 2.4, yields

$$B_n(t) = \sum_{k=0}^{n} \binom{n+1}{k} B_k^{(k-n)}(t).$$

This completes the theorem. □

4 Some approximations to powers of π

The simplest rational appromiations to π are $\frac{22}{7}$ and $\frac{355}{113}$ correction up to two and six decimal places respectively. But, the powers of $22/7$ and $355/113$ does not converge to two and six decimals. For instance

$$\pi^4 - (22/7)^4 = \pi^4 - \frac{234256}{2401} = 0.15692.$$

and

$$\pi^8 - (355/113)^8 = \pi^8 - \frac{15882300625}{163047361} = 0.00645.$$

In this section, some rational approximations to powers of π are given in terms of Bernoulli numbers.

Theorem 4.1. *Let $p_1, p_2, \ldots, p_n$ are first m prime numbers and $n \in N$.*

$$2(2\pi)^{2n} \approx \frac{2.2n!}{|B_{2n}|}\frac{1}{P(2n)} + \frac{4n!}{2n!}\frac{|B_{2n}|}{|B_{4n}|}\frac{1}{Q(2n)}. \tag{4.1}$$

where

$$P(n) = \prod_{k=1}^{m}\left(1-\frac{1}{p_k^n}\right) \tag{4.2}$$

$$Q(n) = \prod_{k=1}^{m}\left(1+\frac{1}{p_k^n}\right) \tag{4.3}$$

Proof. It is well known that from Ref [2, p. 1031]

$$\prod_p\left(1-\frac{1}{p^{2n}}\right) = (-1)^{n-1}\frac{2.2n!}{(2\pi)^{2n}}\frac{1}{B_{2n}}. \tag{4.4}$$

It can be written as follows

$$\prod_{k=m+1}^{\infty}\left(1-\frac{1}{p_k^{2n}}\right) = (-1)^{n-1}\frac{2.2n!}{(2\pi)^{2n}}\frac{1}{B_{2n}}\frac{1}{P(2n)}. \tag{4.5}$$

Similarly,

$$\prod_{k=m+1}^{\infty}\left(1+\frac{1}{p_k^{2n}}\right) = (-1)^{n-1}\frac{4n!}{2n!}\frac{1}{(2\pi)^{2n}}\frac{B_{2n}}{B_{4n}}\frac{1}{Q(2n)}. \tag{4.6}$$

Using (4.5) and (4.6), gives

$$2 \approx \frac{2.2n!}{(2\pi)^{2n}}\frac{1}{|B_{2n}|}\frac{1}{P(2n)} + \frac{4n!}{2n!}\frac{1}{(2\pi)^{2n}}\frac{|B_{2n}|}{|B_{4n}|}\frac{1}{Q(2n)}. \tag{4.7}$$

After simplification, it completes the theorem. □

Example 4.2. The following are some rational approximations to powers of π.

1. Let $n = 2$ and $m = 2$ in (4.1). Then

$$\pi^4 = \frac{1656}{17} \quad \text{and} \quad \pi^4 = \frac{339471}{3485}$$

   gives correction up to 2 and 4 decimals respectively.

2. Let $n = 3$ and $m = 2$ in (4.1). Then

$$\pi^6 = \frac{664320}{691} \quad \text{and} \quad \pi^6 = \frac{4413077316}{4590313}$$

   gives correction up to 4 and 6 decimals respectively.

3. Similarly, for $n = 4$ and $m = 1$

$$\pi^8 = \frac{149944152960}{15802673}$$

   gives correction up 5 decimals.

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