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Sharp Cusa–Huygens and related inequalities

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Abstract: We determine the best positive constants a and b such that

$$\left(\frac{\cos x+2}{3}\right)^a < \frac{\sin x}{x} < \left(\frac{\cos x+2}{3}\right)^b.$$

Similar sharp inequalities are also considered.
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1 Introduction

In paper [4] the author has determined the best positive constants p and q such that

$$\left(\frac{\sinh x}{x}\right)^p < \frac{x}{\sin x} < \left(\frac{\sinh x}{x}\right)^q, \tag{1.1}$$

where $x \in (0, \pi/2)$. In fact one has $p = 1$ and $q \approx 1.18$. Similar results have been obtained in paper [3]:

The best constants $r, s > 0$ such that

$$\frac{1}{(\cosh x)^r} < \frac{\sin x}{x} < \frac{1}{(\cosh x)^s},\ x \in \left(0, \frac{\pi}{2}\right) \tag{1.2}$$

are $r \approx 0.49\ldots$, $s = \dfrac{1}{3}$.

The best constants $u, v > 0$ such that

$$\left(\frac{\sinh x}{x}\right)^u < \frac{2}{\cos x+1} < \left(\frac{\sinh x}{x}\right)^v,\ x \in \left(0, \frac{\pi}{2}\right) \tag{1.3}$$

are $u = 3/2$, $v \approx 1.81\ldots$.

The famous Cusa-Huygens inequality (see e.g. [2]) states that for any $x \in (0, \pi/2)$ one has

$$\frac{\sin x}{x} < \frac{\cos x + 2}{3}. \tag{1.4}$$

As it is well-known that (see e.g. [2])

$$\frac{\sin x}{x} > \frac{\cos x + 1}{2}, \tag{1.5}$$

and as an immediate computation gives

$$\frac{\cos x + 1}{2} > \left(\frac{\cos x + 2}{3}\right)^2$$

(equivalent with $(\cos x - 1)(2\cos x + 1) < 0$), clearly one arises the question on the constants $a, b > 0$ such that

$$\left(\frac{\cos x + 2}{3}\right)^a < \frac{\sin x}{x} < \left(\frac{\cos x + 2}{3}\right)^b. \tag{1.6}$$

Similarly, as it is shown in [2], one has

$$\frac{\sin x}{x} > \left(\frac{\cos x + 1}{2}\right)^{2/3},$$

by (1.5) we can study the constants c and $d > 0$ such that

$$\left(\frac{\cos x + 1}{2}\right)^c < \frac{\sin x}{x} < \left(\frac{\cos x + 1}{2}\right)^d, \tag{1.7}$$

where, as in the case of (1.6), $x \in (0, \pi/2)$.

The hyperbolic variants of these inequalities may be studied, too. In what follows, we shall always assume that $x \in (0, \pi/2)$.

2 Main results

First, by using the method of [4] we shall prove the following:

Theorem 2.1. The best positive constants a and b in inequality (1.6) are $a = (\ln \pi/2)/(\ln 3/2) \approx 1,113\ldots$ and $b = 1$.

Proof. We shall use the following auxiliary results:

Lemma 2.1. One has, for any $x \in (0, \pi/2)$, the inequalities

$$\ln \frac{x}{\sin x} < \frac{\sin x - x\cos x}{2\sin x} \tag{2.1}$$

and

$$\ln \frac{3}{2 + \cos x} > \frac{x \sin x}{2(2 + \cos x)}. \tag{2.2}$$

Proof. Inequality (2.1) is proved in [4] (see Lemma 2.2). For the proof of (2.2) consider the application

$$a(x) = \ln \frac{3}{2 + \cos x} - \frac{x \sin x}{2(2 + \cos x)}, \quad x \in \left[0, \frac{\pi}{2}\right).$$

For the derivative of this function one can deduce, by an elementary computation:

$$2(2+\cos x)^2 a'(x) = 2\sin x + \sin x\cos x - 2x\cos x - x = b(x).$$

Now $b'(x) = 2\sin x(x - \sin x) > 0$, and as $b(0) = 0$, we get $b(x) \geq b(0) = 0$ for $x \geq 0$. This in turn implies $a'(x) \geq 0$ for $x \geq 0$ and as $a(0) = 0$, we get $a(x) > 0$ for $x > 0$ and $x < \pi/2$. This proves relation (2.2) of Lemma 2.1.

Proof of Theorem 2.1. Let us introduce the application

$$h(x) = \frac{\ln(x/\sin x)}{\ln(3/(2+\cos x))},\ x \in \left(0, \frac{\pi}{2}\right)$$

and

$$f(x) = \ln(x/\sin x),\ g(x) = \ln(3/(2+\cos x)).$$

One gets easily

$$g^2(x)h'(x) = \frac{\sin x - x\cos x}{x\sin x}\ln\frac{3}{2+\cos x} - \frac{\sin x}{2+\cos x}\ln\frac{x}{\sin x}. \tag{2.3}$$

By inequality (2.1) one can write

$$g^2(x)h'(x) > \frac{\sin x - x\cos x}{\sin x}\left[\frac{1}{x}\ln\frac{3}{2+\cos x} - \frac{\sin x}{2(2+\cos x)}\right],$$

so by (2.2), the paranthesis being strictly positive, we get by (2.3)

$$h'(x) > 0.$$

Thus $h(x)$ is a strictly increasing function. This implies

$$\lim_{x\to 0} h(x) = 1 < h(x) < h\left(\frac{\pi}{2}\right) = \frac{\ln \pi/2}{\ln 3/2},$$

so we get the best constants in (1.6), $a = \dfrac{\ln \pi/2}{\ln 3/2} \approx 1,113\ldots$ and $b = 1$. This proves Theorem 2.1.

In what follows, we shall prove by another method the following result:

Theorem 2.2. The best positive constants c and d in inequality (1.7) are $c = \dfrac{2}{3}$ and

$$d = \frac{\ln(\pi/2)}{\ln 2} \approx 0.651\ldots.$$

Proof. The following variant of L'Hôpital's rule, known also as the "monotone form of L'Hôpital's rule" will be applied (see [1], p. 106):

Lemma 2.2. For $a < b$, let f, g be continuous on $[a, b]$, differentiable on (a, b) and g' never vanish on (a, b). If f'/g' is (strictly) increasing (decreasing) on (a, b), then so are $\dfrac{f(x) - f(a)}{g(x) - g(a)}$ and $\dfrac{f(x) - f(b)}{g(x) - g(b)}$.

Let $f(x) = \ln \dfrac{2}{\cos x + 1}$ and $g(x) = \ln \dfrac{x}{\sin x}$, where $[a, b] = [0, \pi/2]$. Then,

$$\frac{f'(x)}{g'(x)} = \frac{x \sin^2 x}{(\sin x - x\cos x)(\cos x + 1)} = \frac{2x \sin^2 \dfrac{x}{2}}{\sin x - x\cos x} = \frac{f_1(x)}{g_1(x)}.$$

One has

$$\frac{f_1'(x)}{g_1'(x)} = \frac{2\sin \dfrac{x}{2}\left(\sin \dfrac{x}{2} + x\cos \dfrac{x}{2}\right)}{x \sin x} = 1 + \frac{\tan \dfrac{x}{2}}{x}.$$

As for $k(x) = \dfrac{\tan \frac{x}{2}}{x}$ one has $k'(x) = \dfrac{x - \sin x}{2\cos^2 \frac{x}{2}} > 0$, the function $k(x)$ is strictly increasing. As $f_1(0) = g_1(0) = 0$, $\dfrac{f_1(x)}{g_1(x)}$ will be strictly increasing. This in turn implies the same for the function $\dfrac{f(x)}{g(x)} = h(x)$. As (1.7) may be written as $\dfrac{1}{c} < h(x) < \dfrac{1}{d}$, and as $h(x)$ is strictly increasing, we get

$$\frac{1}{c} = \lim_{x\to 0} h(x) = \frac{3}{2},\ \frac{1}{d} = h\left(\frac{\pi}{2}\right) = \frac{\ln 2}{\ln(\pi/2)}.$$

This proves Theorem 2.2.

There exist also hyperbolic variants to these theorems. We prove one of these theorems, namely:

Theorem 2.3. The best positive constants m and n such that

$$\left(\frac{\cosh x + 1}{2}\right)^m < \frac{\sinh x}{x} < \left(\frac{\cosh x + 1}{2}\right)^n,\ x > 0$$

are $m = \dfrac{2}{3}$ and $n = 1$.

Proof. As in the proof of Theorem 2.2, let

$$h(x) = \ln\left(\frac{\sinh x}{x}\right) / \ln\left(\frac{\cosh x + 1}{2}\right) = f(x)/g(x),\ x \in (0, +\infty).$$

Then, by elementary computations we get

$$\frac{f'(x)}{g'(x)} = \frac{x\cosh x - \sinh x}{2x \sinh^2 \dfrac{x}{2}} = \frac{f_1(x)}{g_1(x)}.$$

One gets

$$\frac{f_1'(x)}{g_1'(x)} = \frac{x \sinh x}{2\sinh \dfrac{x}{2}\left(\sinh \dfrac{x}{2} + x\cosh \dfrac{x}{2}\right)} = \frac{1}{1 + \left(\tanh \dfrac{x}{2}\right)/x}.$$

Put $l(x) = \dfrac{\tanh \frac{x}{2}}{x}$. As $l'(x) = \dfrac{x - \sinh x}{2\cosh^2 \frac{x}{2}} < 0$, $l(x)$ is strictly decreasing for any $x > 0$, so $\dfrac{f_1'(x)}{g_1'(x)}$ is strictly increasing. As $f_1'(0) = g_1'(0) = 0$ and $f(0) = g(0) = 0$, $\dfrac{f(x)}{g(x)} = h(x)$ will be strictly increasing on $(0, b)$ for any $b > 0$. Thus, we get from Lemma 2.2 that $h(x)$ is strictly increasing for any $x \in (0, b)$; thus

$$\lim_{x\to 0} h(x) = \frac{2}{3} < h(x) < h(b) = \ln \frac{\sinh b}{b} / \ln \left(\frac{\cosh b + 1}{2} \right).$$

As $\lim_{b\to +\infty} h(b) = 1$, we get $m = \frac{2}{3}$ and $n = 1$, so Theorem 2.3 follows.

References

[1] Hardy, G.H., J.E. Littlewood, G. Pólya. *Inequalities,* Cambridge Univ. Press, 1959.

[2] Neuman, E., J. Sándor. On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker, and Huygens inequalities, *Math. Ineq. Appl.,* Vol. 13, 2010, No. 4, 715–723.

[3] Neuman, E., J. Sándor. Optimal inequalities for hyperbolic and trigonometric functions, *Bull. Math. Anal. Appl.*, Vol. 3, 2011, No. 3, 177–181.

[4] Sándor, J. Two sharp inequalities for trigonometric and hyperbolic functions, *ath. Ineq. Appl.*, to appear

Notes added in proof

This paper was written in 2010, and sent to the journal September 19, 2011.

Meantime, paper [4] (sent May 9, 2011) has been published in Vol. 15, 2012, No. 2, 409–413. A Referee has pointed out that, Theorem 2.1 of this paper has been discovered also in the following work: C.-P. Chen and W.-S. Cheung, *Sharp Cusa and Becker–Stark inequalities*, J. Ineq. Appl., 2011:136.

As one can see, our method is based on the earlier paper [4], while the above work uses completely different (and more complicated) arguments. As this paper appeared 7 December 2011, clearly the result has been sent to journals about the same times independently.