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New explicit formulae for the prime counting function

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Abstract: In the paper new explicit formulae for the prime counting function π are proposed and proved. They depend on arbitrary positive arithmetic function which satisfies certain condition. As a particular case a formula for π depending on Euler's function φ is obtained. To the author's best knowledge such kind of formulae are proposed for the first time in the mathematical literature.
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Used denotations

$\lfloor\rfloor$ – denotes the floor function, i.e. $\lfloor x \rfloor$ denotes the largest integer that is not greater than the real non-negative number x; $\prod_{p|n}$ – denotes that the product is taken over all prime divisors of the integer $n > 1$; φ – denotes Euler's totient function, i.e. $\varphi(1) = 1$ and for integer $n > 1$

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right);$$

ψ – denotes Dedekind's function, i.e. $\psi(1) = 1$ and for integer $n > 1$

$$\psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p}\right);$$

σ – denotes the so-called sum-of-divisors function, i.e. $\sigma(1) = 1$ and for integer $n > 1$

$$\sigma(n) = \sum_{d|n} d,$$

where $\sum\limits_{d|n}$ means that the sum is taken over all divisors d of n; ζ – denotes Riemman's zeta function.

1 Introduction

Let $n \geq 2$ be an integer. As usually, the prime counting function π for a given integer $n \geq 2$ denotes the number of primes p, satisfying the inequality $p \leq n$.

There are many known explicit formulae for $\pi(n)$. The first example are Willans formulae ([5, p.180]):

$$\pi(n) = -1 + \sum_{j=1}^{n} F(j),$$

where $F(j) = \left\lfloor \cos^2\left(\pi \frac{(j-1)!+1}{j}\right)\right\rfloor$;

$$\pi(n) = \sum_{j=2}^{n} H(j),$$

where $H(j) = \left(\sin^2\left(\frac{\pi}{j}\right)\right)^{-1} \sin^2\left(\pi \frac{((j-1)!)^2}{j}\right)$.

The second example is Mináč formula ([5, p.181]):

$$\pi(n) = \sum_{j=2}^{n} \left\lfloor \frac{(j-1)!+1}{j} - \left\lfloor \frac{(j-1)!}{j} \right\rfloor \right\rfloor.$$

The third example is the formula contained in [6, p.414]:

$$\pi(n) = -1 + \sum_{j=3}^{n} \left\lfloor (j-2)! - j \left\lfloor \frac{(j-2)!}{j} \right\rfloor \right\rfloor.$$

Other type of representations for function π were proposed by K. Atanassov in [1–3] and by K. Atanassov and M. Vassilev-Missana in [4].

But all of these formulae share the same disadvantage – they represent $\pi(n)$ as a sum of terms that are integers or which are expressed with the help of the floor function or similar functions.

Many years ago, the author had also proposed such kind of formulae in [7]:

$$\pi(n) = -2 \sum_{k=2}^{n} \zeta(-2(k-1-\varphi(k)));$$

$$\pi(n) = -2 \sum_{k=2}^{n} \zeta(-2(\sigma(k)-k-1));$$

$$\pi(n) = \sum_{k=2}^{n} \left\lfloor \frac{1}{k-\varphi(k)} \right\rfloor;$$

$$\pi(n) = \sum_{k=2}^{n} \left\lfloor \frac{1}{\sigma(k) - k} \right\rfloor;$$

$$\pi(n) = \sum_{k=2}^{n} \left\lfloor \frac{k+1}{\sigma(k)} \right\rfloor;$$

$$\pi(n) = \sum_{k=2}^{n} \left\lfloor \frac{\varphi(k)}{k-1} \right\rfloor. \tag{1}$$

One may easily observe also the validity of the formulae:

$$\pi(n) = -2 \sum_{k=2}^{n} \zeta(-2(\psi(k) - k - 1));$$

$$\pi(n) = \sum_{k=2}^{n} \left\lfloor \frac{1}{\psi(k) - k} \right\rfloor;$$

$$\pi(n) = \sum_{k=2}^{n} \left\lfloor \frac{k+1}{\psi(k)} \right\rfloor.$$

Formula (1) is contained also in [8, p. 181]. It suggests other type of explicit formulae for $\pi(n)$.

It is important to note that these formulae produce $\pi(n)$ as a part of an infinite series, depending on arbitrary arithmetic function with strictly positive values (satisfying certain condition) and more precisely as a number (depending on n), the integer part of which coincides with $\pi(n)$. To the author's best knowledge such approach is proposed for the first time.

In this way the fundamental shortcoming of the mentioned above formulae is overcome.

2 Main results

First we need the following two auxilliary results (the first due to Sierpiński):

Lemma 1 ([9, p.248, Theorem 5]). *Let $k > 1$ be a composite number. Then the inequality*

$$\varphi(k) \le k - \sqrt{k}$$

holds. Equality is possible only when $k = p^2$, with p – prime number.

Corollary 1. *Under the conditions of Lemma 1 it is fulfilled:*

$$\frac{\varphi(k)}{k-1} \le \frac{k - \sqrt{k}}{k-1} = 1 - \frac{1}{\sqrt{k}+1}. \tag{2}$$

The following important theorem provides infinitely many examples of explicit formulae for the prime counting function π of the discussed type.

Theorem 1. *Let f be an arithmetic function with strictly positive values and let there exist a composite number $T(f) > 1$, such that the inequality*

$$\sum^{*}\left(\frac{\varphi(k)}{k-1}\right)^{f(k)} + \sum_{k=T(f)}^{\infty} e^{-\frac{f(k)}{\sqrt{k}+1}} < 1 \qquad (3)$$

holds (where the symbol $\sum^{*}$ is used for the summation over all composite numbers k, satisfying: $4 \le k \le T(f) - 1$). Then for any integer $n \ge 2$

$$\pi(n) = \left\lfloor \sum_{k=2}^{n} \left(\frac{\varphi(k)}{k-1}\right)^{f(k)} \right\rfloor. \qquad (4)$$

Remark 1. Further we suppose that $T(f)$ is the minimal composite number satisfying (3).

Remark 2. For $T(f) = 4$, (3) is reduced to the condition

$$\sum_{k=T(f)}^{\infty} e^{-\frac{f(k)}{\sqrt{k}+1}} < 1.$$

Proof of the theorem. Let the integer $n \ge 2$ be arbitrarily chosen. We have

$$\sum_{k=2}^{n} \left(\frac{\varphi(k)}{k-1}\right)^{f(k)} = \sum\nolimits_{1} \left(\frac{\varphi(k)}{k-1}\right)^{f(k)} + \sum\nolimits_{2} \left(\frac{\varphi(k)}{k-1}\right)^{f(k)}, \qquad (5)$$

where the symbol $\sum_1$ is used for the summation over all primes k, satisfying the inequality $2 \le k \le n$ and the symbol $\sum_2$ is used for the summation over all composite numbers satisfying the same inequality.

Since $\varphi(k) = k - 1$ (for prime k), we obtain

$$\sum\nolimits_{1} \left(\frac{\varphi(k)}{k-1}\right)^{f(k)} = \pi(n). \qquad (6)$$

Let $n \le 3$. Then $\sum_2$ does not exist and (4) follows immediately from (6) (without using the floor function and without the need of condition (3)).

Let $T(f)$ is from the condition of Theorem 1 and let $n \ge 4$. If $n < T(f)$, then (3) implies

$$\sum\nolimits_{2} \left(\frac{\varphi(k)}{k-1}\right)^{f(k)} < 1. \qquad (7)$$

Hence (4) holds again.

Let $n \ge T(f)$. Then we have

$$\sum\nolimits_{2} \left(\frac{\varphi(k)}{k-1}\right)^{f(k)} = \sum^{*} \left(\frac{\varphi(k)}{k-1}\right)^{f(k)} + \sum\nolimits_{2}' \left(\frac{\varphi(k)}{k-1}\right)^{f(k)}, \qquad (8)$$

where the symbol $\sum_2'$ is used for the summation over all composite k, such that $T(f) \le k \le n$.

Let $\{a_k\}_{k=1}^{\infty}$ be the sequence defined by:

$$a_k \stackrel{\text{def}}{=} \left(1 - \frac{1}{\sqrt{k}+1}\right)^{\sqrt{k}+1}.$$

Then (2) yields

$$\sum\nolimits_2' \left(\frac{\varphi(k)}{k-1}\right)^{f(k)} \leq \sum\nolimits_2' a_k^{\frac{f(k)}{\sqrt{k}+1}} < \sum_{k=T(f)}^{\infty} a_k^{\frac{f(k)}{\sqrt{k}+1}}. \tag{9}$$

But it is known that $\{a_k\}_{k=1}^{\infty}$ is a strictly monotonously increasing sequence and

$$\lim_{k\to\infty} a_k = e^{-1}.$$

From this and from (9) we obtain

$$\sum\nolimits_2' \left(\frac{\varphi(k)}{k-1}\right)^{f(k)} < \sum_{k=T(f)}^{\infty} e^{-\frac{f(k)}{\sqrt{k}+1}}. \tag{10}$$

Now (3), (8) , (9) and (10) imply (7). Therefore, from (5), (6) and (7), the equality (4) holds and the theorem is proved. □

The following result is a corollary from Theorem 1. It represents our main formula for the prime counting function π.

Theorem 2. *Let $n \geq 2$ be an integer. Then the explicit formula:*

$$\pi(n) = \left\lfloor \sum_{k=2}^{n} \left(\frac{\varphi(k)}{k-1}\right)^{k-1} \right\rfloor \tag{11}$$

is valid.

Proof. Let the integer $n \geq 2$ be arbitrarily chosen. We put $f(k) = k-1,\ k = 2, 3, 4, \ldots$. Now we must verify that for $T(f) = 20$, the condition (3) is satisfied. In this case we have:

$$\begin{aligned}\sum\nolimits^{*} \left(\frac{\varphi(k)}{k-1}\right)^{k-1} &= \frac{(\varphi(4))^3}{3^3} + \frac{(\varphi(6))^5}{5^5} + \frac{(\varphi(8))^7}{7^7} + \frac{(\varphi(9))^8}{8^8} + \frac{(\varphi(10))^9}{9^9} + \frac{(\varphi(12))^{11}}{11^{11}} + \\ &+ \frac{(\varphi(14))^{13}}{13^{13}} + \frac{(\varphi(15))^{14}}{14^{14}} + \frac{(\varphi(16))^{15}}{15^{15}} + \frac{(\varphi(18))^{17}}{17^{17}} = \\ &= \frac{16069204238009440618274884533684317607038075807395555829725998015338552938555645811}{3756642962600779314817302855472169007529339855960503129361183318150200000000000000000} \\ &= 0.427754\ldots < 0.43. \end{aligned} \tag{12}$$

Since $\frac{k-1}{\sqrt{k}+1} = \sqrt{k} - 1,\ k = 20, 21, 22, \ldots$, we have:

$$\sum_{k=20}^{\infty} e^{-\frac{k-1}{\sqrt{k}+1}} = \sum_{k=20}^{\infty} e^{-(\sqrt{k}-1)} = e \sum_{k=20}^{\infty} e^{-\sqrt{k}}. \tag{13}$$

Since the function $g(k) = e^{-\sqrt{k}}$ is strictly monotonously decreasing in the interval $[19, +\infty)$, we have:

$$\sum_{k=20}^{\infty} g(k) < \int_{19}^{+\infty} g(k)\, dk,$$

i.e.

$$\sum_{k=20}^{\infty} e^{-\sqrt{k}} < \int_{19}^{+\infty} e^{-\sqrt{k}}\, dk = 2\int_{\sqrt{19}}^{+\infty} te^{-t}\, dt = 2(1+\sqrt{19})e^{-\sqrt{19}}. \tag{14}$$

From (13) and (14) we obtain:

$$\sum_{k=20}^{\infty} e^{-\frac{k-1}{\sqrt{k}+1}} < 2(1+\sqrt{19})e^{1-\sqrt{19}} = 0.37269\ldots < 0.38. \tag{15}$$

Now (12) and (15) yield:

$$\sum\nolimits^{*} \left(\frac{\varphi(k)}{k-1}\right)^{k-1} + \sum_{k=20}^{\infty} e^{-\frac{k-1}{\sqrt{k}+1}} < 0.43 + 0.38 = 0.81 < 1. \tag{16}$$

But (16) means that the condition (3) is satisfied for $f(k) = k-1$ $(k = 2, 3, 4, \ldots)$. Therefore, applying Theorem 1, we conclude that formula (11) is valid. □

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