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On the composition of the functions σ and φ on the set $Z_s^+(P^*)$

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Abstract: In 1964, A. Mąkowski and A. Schinzel ([8],Cf.[6]) conjectured that for all positive integers m, we have

$$\frac{\sigma(\varphi(m))}{m} \geq \frac{1}{2}, \tag{*}$$

where σ denote the sum of divisors function and φ is the Euler's totient function.

Let P be the set of all odd primes and

$$P^* = \{p \in P; p = 2^\alpha k + 1, \alpha \geq 1, k > 1, (k, 2) = 1\}.$$

Moreover, let

$$Z_s^+(P^*) = \left\{ n = \prod_{j=1}^{r} p_j; p_j = 2^{\alpha_j} m_j + 1; \alpha_j \geq 1, m_j > 1, p_j \in P^* \right\} \tag{**}$$

where $(m_j, m_k) = 1$; for all $j \neq k$; $j, k = 1, 2, ..., r.$

In this paper we prove that if $n \in Z_s^+(P^*)$ then we have $\frac{\sigma(\varphi(n))}{n} \geq 1$. From this and Sandor's result it follows that (*) is true for all positive integers $m \geq 1$ such that the squarefree part of m $\in Z_s^+(P^*)$.

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1 Introduction

Interesting results about the Mąkowski–Schinzel's conjecture given in (*) has been proved by many authors.

Inequality (*) was first investigated by J. Sandor (see [10–12]). M. Filaseta, S. W. Graham and C. Nicol in the paper [3] verified the inequality (*) for $n = p_1 \cdot p_2 \cdot ... \cdot p_k$ where p_i denotes the i-th prime number.

U. Balakrishnan [1] proved the same for all squarefull numbers. In the paper [4] A. Grytczuk, F. Luca and M. Wójtowicz proved that the lower density of the set of integers satisfying the inequality (*) is greater than 0.74.

Using sieve techniques K. Ford [4] proved that for all positive integers m we have $\frac{\sigma(\varphi(m))}{m} > \frac{1}{39.4}$. Further results can be found in the paper by F. Luca and C. Pommerance [7].

Many other interesting and important results concerning of the Mąkowski-Schinzel conjecture and the Euler's function has been given in very nice monograph "Handbook of Number Theory, II" by J. Sandor and B. Crstici, (see [13, Chapter 3, p.4]) and Sandor's papers: [10–12] and [14].

J.Sandor [11, 12] proved that (*) is true if and only if is true for all squarefree positive integers.This fact has been rediscovered by G. L. Cohen [2].

Using some techniques from the Sandor's papers we prove of the following theorem:

Theorem. *For all positive integers $n \in Z_s^+(P^*)$ we have*

$$\frac{\sigma(\varphi(n))}{n} \geq 1. \tag{***}$$

Immediately from the Theorem follows the following Corollary;

Corollary. *If $m = 2n$, where $n \in Z_s^+(P^*)$, then we have*

$$\frac{\sigma(\varphi(m))}{m} \geq \frac{1}{2}.$$

2 Basic lemmas

Lemma 1 (Langford's inequality, (see [9, p. 434]))**.** *Let σ be the sum of divisors function and τ be the function of all divisors of the positive integer $n \geq 2$.Then we have*

$$\sigma(n) \geq n + 1 + \sqrt{n}(\tau(n) - 2). \tag{2.1}$$

Lemma 2 (Sandor's inequalities, (see [10], Lemma 1))**.** *Let σ be the sum of divisors function. Then for all positive integers m, n we have*

$$\sigma(mn) \geq m\sigma(n), \tag{2.2}$$

and for all positive integers m, n such that there is at least one prime q such that $q \mid m$ and $q \nmid n$, we have

$$\sigma(mn) \geq (m+1)\sigma(n). \tag{2.3}$$

3 Proof of the Theorem

Let $n \in Z_s^+(P^*)$. Then we have $n = p_1 p_2 ... p_k$, where p_j are odd primes of the form

$$p_j = 2^{\alpha_j} m_j + 1; (2, m_j) = 1; m_j > 1, \alpha_j \geq 1; j = 1, 2, ..., k. \quad (3.1)$$

By the well-known property of the Euler's totient function φ and (3.1) it follows that

$$\varphi(n) = \varphi(p_1 p_2 ... p_k) = (p_1 - 1) ... (p_k - 1) = 2^{\alpha_1 + \alpha_2 + ... + \alpha_k} m_1 m_2 ... m_k. \quad (3.2)$$

From (3.1), (3.2) and the multiplicative property of the function σ we obtain

$$\sigma(\varphi(n)) = (2^{\alpha_1 + \alpha_2 + ... + \alpha_k + 1} - 1) \sigma(m_1 m_2 ... m_k). \quad (3.3)$$

We prove the inequality (***) of the Theorem by induction with respect to $k = \omega(n)$, where $\omega(n)$ denotes the number of distinct primes in the number $n \in Z_s^+(P^*)$.

For $k = 1$,we have $p_1 = 2^{\alpha_1} m_1 + 1$ and (***) immediately follows from Lemma 1. Suppose that (***) holds for all $n \in Z_s^+(P^*)$ with $k \leq r$.Now , let $m \in Z_s^+(P^*)$ and $k = r + 1$, so

$$m = p_1 p_2 ... p_r p_{r+!}, p_j = (2^{\alpha_1} m_1 + 1)(2^{\alpha_2} m_2 + 1) ... (2^{\alpha_r} m_r + 1)(2^{\alpha_{r+1}} m_{r+1} + 1). \quad (3.4)$$

From (3.4) and the property of the Euler's function φ we get

$$\varphi(m) = (p_1 - 1)(p_2 - 1) ... (p_{r+1} - 1) = 2^{\alpha_1 + \alpha_2 + ... + \alpha_r + \alpha_{r+1}} m_1 m_2 ... m_r m_{r+1}. \quad (3.5)$$

By (3.5) and the fact that $(2, m_j) = 1, j = 1, 2, ...r, r + 1$ and multiplicative property of the function σ it follows that

$$\sigma(\varphi(m)) = \sigma\left(2^{\alpha_1 + \alpha_2 + ... + \alpha_r + \alpha_{r+1}}\right) \sigma(m_1 m_2 ... m_r m_{r+1}). \quad (3.6)$$

From (3.6) and the well known formula on the function σ it follows that

$$\sigma(\varphi(m)) = \left(2^{\alpha_1 + \alpha_2 + ... + \alpha_r + \alpha_{r+1} + 1} - 1\right) \sigma(m_1 m_2 ... m_r m_{r+1}). \quad (3.7)$$

By the definition of the set $Z_s^+(P^*)$ it follows that $(m_i, m_j) = 1$ for all distinct $i, j = 1, 2, ..., r, r + 1$; hence we have

$$(m_1 m_2 ... m_r; m_{r+1}) = 1. \quad (3.8)$$

From (3.7),(3.8) and the multiplicative property of the function σ we obtain

$$\sigma(\varphi(m)) = \left(2^{\alpha_1 + \alpha_2 + ... + \alpha_r + \alpha_{r+1} + 1} - 1\right) \sigma(m_1 m_2 ... m_r) \sigma(m_{r+1}). \quad (3.9)$$

Now, we note that inductive assumption implies

$$\frac{\sigma\left(\varphi\left(p_1p_2...p_r\right)\right)}{p_1p_2...p_r} = \frac{2^{\alpha_1+\alpha_2+...+\alpha_r+1}-1}{\left(2^{\alpha_1}m_1+1\right)\left(2^{\alpha_2}m_2+1\right)...\left(2^{\alpha_r}m_r+1\right)}\sigma\left(m_1m_2...m_r\right) \geq 1. \tag{3.10}$$

Since $m_{r+1} \geq 2$ then by (3.9), (3.10) and Lemma 2 it follows that

$$\frac{\sigma\left(\varphi\left(m\right)\right)}{m} \geq \frac{2^{\alpha_1+...+\alpha_r+\alpha_{r+1}+1}-1}{\left(2^{\alpha_1}m_1+1\right)...\left(2^{\alpha_r}m_r+1\right)\left(2^{\alpha_{r+1}}m_{r+1}+1\right)} \frac{\left(2^{\alpha_1}m_1+1\right)...\left(2^{\alpha_r}m_r+1\right)}{2^{\alpha_1+...+\alpha_r+1}-1}\left(m_{r+1}+1\right). \tag{3.11}$$

By (3.11) it follows that

$$\frac{\sigma\left(\varphi\left(m\right)\right)}{m} \geq \frac{\left(2^{\alpha_1+...+\alpha_r+\alpha_{r+1}+1}-1\right)\left(m_{r+1}+1\right)}{\left(2^{\alpha_1+...+\alpha_r+1}-1\right)\left(2^{\alpha_{r+1}}m_{r+1}+1\right)} \geq 1, \tag{3.12}$$

and the proof of the Theorem is complete. □

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