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A comment on a result of Virgolici

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Abstract: We provide a short proof of a generalization of a recent result of Virgolici on the diophantine equation $2^x + 1009^y = p^z$.
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1 Introduction

In a recent paper, Virgolici [3] solved the exponential equation

$$2^x + 1009^y = p^z \tag{1}$$

for odd primes $p < 1000$, showing that there are no solutions with $y > 0$ (and only solutions with $y = 0$ for the Fermat primes $p = 3, 5, 17$ and 257).

2 Main result

In this note, we show more generally the following.

Theorem 1 *If there exists a solution to equation (1) in integers x, y and z and prime p, then either $(x, y, z, p) = (3, 0, 2, 3)$ or $z = 1$.*

To prove this, we will assume that we have a solution to (1) with, via Mihailescu's Theorem, $y > 0$. If z is even, then necessarily $p = 3$, as treated in [3]. Otherwise, we may suppose that z is odd and that, modulo 9, x is even. if $3 \mid z$, applying Magma's SIntegralPoints routine (here $S = \{1009\}$) to the elliptic curves $Y^2 = X^3 - 1009^\alpha$ for $0 \leq \alpha \leq 5$, we find no new solutions. If $5 \mid z$, equation (1) is insoluble modulo

$$11 \cdot 31 \cdot 41 \cdot 61 \cdot 251 \cdot 331 \cdot 401 \cdot 601.$$

We may thus assume that z is divisible by a prime $q \geq 7$. Equation (1) is now a special case of the equation $a^2 + 1009^y = b^q$, where b is odd and $q \geq 7$ is prime. Appealing to the Primitive Divisor Theorem of Bilu et al [1] in a now-standard way (as in, for example, [2]) and using the fact that $\mathbb{Q}(\sqrt{-1009})$ has class number 20, we conclude that there are no solutions to $a^2 + 1009^y = b^q$ with $q \geq 7$. This completes the proof of Theorem 1.

Note that there are many solutions to (1) with $z = 1$, likely infinitely many. These are of course just primes of the shape $2^x + 1009^y$. There are precisely 20 such primes up to 10^{20}.

It is not too hard to generalize Theorem 1 by replacing p with an arbitrary integer. This adds the additional solution corresponding to the identity $2^9 + 1009 = 39^2$. Our arguments also work with minor modification if one replaces the prime 1009 with any prime congruent to one of $1, 13$ or 19 modulo 24.

References

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