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Solving algebraic equations with Integer Structure Analysis

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Abstract: A new alternative method for solving algebraic equations is expounded. Integer Structure Analysis is used with an emphasis on parity, right-end-digits of the components and the modular ring Z_5.

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1 Introduction

Many methods exist for finding real and complex solutions of algebraic equations such as quadratic, cubic and simultaneous equations with two and three variables [1, 3]. Integer Structure Analysis (ISA) can be used with these traditional methods to illuminate some of the related underlying number theoretic foundations [6]. In this paper, we provide examples of various types of equations using the modular ring Z_5 and right-end-digits (REDs) (Table 1). In this ring each class has a characteristic RED structure that simplifies analysis.

Row	$f(r)$	$5r_0$	$5r_1+1$	$5r_2+2$	$5r_3+3$	$5r_4+4$
	Class	$\bar{0}_5$	$\bar{1}_5$	$\bar{2}_5$	$\bar{3}_5$	$\bar{4}_5$
0		0	1	2	3	4
1		5	6	7	8	9
2		10	11	12	13	14
3		15	16	17	18	19
4		20	21	22	23	24

(table continues)

Row	$f(r)$	$5r_0$	$5r_1+1$	$5r_2+2$	$5r_3+3$	$5r_4+4$
	Class	$\bar{0}_5$	$\bar{1}_5$	$\bar{2}_5$	$\bar{3}_5$	$\bar{4}_5$
5		25	26	27	28	29
6		30	31	32	33	34
7		35	36	37	38	39
8		40	41	42	43	44
9		45	46	47	48	49
10		50	51	52	53	54

Table 1. Rows of Z_5

2 Quadratic equations

In the following sections we solve some simple algebraic equations with an ISA approach to show how it sheds some light on the underlying number theoretic structure.

1.
$$5x^2 - 17x + 14 = 0. \tag{2.1}$$

If x is odd then $x^* = 7$, and if x is even, then $x^* = 2$, so

$$x = 5r_2 + 2. \tag{2.2}$$

Substituting (2.2) into (2.1) yields

$$25r_2^2 - 3r_2 = 0. \tag{2.3}$$

Thus $r_2 = 0$ or $-3/25$, and so $x = 2$ or $7/5$.

2.
$$f(x) \equiv x^2 - 11x + 30 = 0. \tag{2.4}$$

x can be odd or even.

x^*	1	3	5	7	9	x^*	0	2	4	6	8
$(x^2)^*$	1	9	5	9	1	$(x^2)^*$	0	4	6	6	4
$(-11x)^*$	-1	-3	-5	-7	-9	$(-11x)^*$	0	-2	-4	-6	-8
$(30)^*$	0	0	0	0	0	$(30)^*$	0	0	0	0	0
$(f(x))^*$	0	6	0	2	8	$(f(x))^*$	0	2	2	0	6

Table 2: Solution structure for Equation (2.4)

Thus $x^* = 1$ or 5 and $x^* = 0$ or 6, but $x \neq 1$ or 0, and so $x = 5$ or 6. Alternatively, use Z_5, take $x = 5r + a$ and substitute into (2.4). If $a = 6$, the constant 30 is eliminated and we obtain

$$25r^2 + 5r = 0. \tag{2.5}$$

Thus, $r = 0$ or $-1/5$ which gives $x = 6$ or 5.

3 Cubic equations

1. $$x^3 + 4x - 5 = 0. \tag{3.1}$$

x must be odd.

x*	1	3	5	7	9
(x^3)*	1	7	5	3	9
$(4x)$*	4	2	0	8	6
(-5)*	–5	–5	–5	–5	–5
$(f(x))$*	0	4	0	6	0

Table 3: Structure for Equation (3.1)

Thus $x^* = 1$, 5 or 9, but 5 and 9 are too large, so $x^* = 1$, and

$$x = 5r_1 + 1. \tag{3.2}$$

Substituting into (3.1) yields $r_1 = 0$ or

$$25r_1^2 + 15r_1 + 7 = 0. \tag{3.3}$$

Then using $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, we get

$$r_1 = \frac{-3 \pm \sqrt{19}i}{10} \tag{3.4}$$

so that

$$x = 1, \frac{-1 \pm \sqrt{19}i}{2}$$

2. $$x^3 + 5x^2 + 3x - 9 = 0. \tag{3.5}$$

x*	1	3	5	7	9
$(5x^2)$*	5	5	5	5	5
$(3x)$*	±3	±9	±5	±1	±7
(x^3)*	±1	±7	±5	±3	±9
$(-q)$*	–9	–9	–9	–9	–9
$(f(x))$*	0	2	6	0	2

Table 4: Structure for Equation (3.5)

From Table 4, $x^* = 1$ or 7, but 7 is too large so

$$x = 5r_1 + 1. \tag{3.6}$$

Substituting into (3.5) gives $r_1 = 0$ and

$$25r_1^2 + 40r_1 + 16 = 0. \tag{3.7}$$

Thus $r_1 = 0$ or $-4/5$, and so $x = 1$ or -3. When x is negative, $x^* = -3$, and so $x = 1, -3, -3$. For other non-conventional number theoretic approaches to cubic equations see [2, 4, 5, 6].

4 Simultaneous equations

4.1 Two variables

1.
$$3x + 7y = 27 \quad (4.1)$$
$$5x + 2y = 16 \quad (4.2)$$

x must be even from (4.2), so from (4.1) y must be odd. Since $(5x)^* = 0$ and y is odd, only $y^* = 3$ or $(2y)^* = 6$. Thus $y = 3, 13, 23, \ldots$ but only $y = 3$ fits, so that the solution is

$$x = 2, y = 3.$$

2.
$$x^2 + 4y^2 + 80 = 15x + 30y \quad (4.3)$$
$$xy = 6 \quad (4.4)$$

From (4.4) $x = 6/y$ which, when substituted into (4.3), gives

$$4y^4 - 30y^3 + 80y^2 - 90y + 36=0 \quad (4.5)$$

The RED right hand side is zero, so $(4y^4 + 36)^* = 0$. Hence, $y^* =1$, 2 or 3 and $y = 1, 2, 3$ which fits (4.5) and so yields $x = 6, 3, 2$. But Equation (4.3) should have 4 roots, so we substitute (4.4) into (4.3) to get

$$x^4 - 15x^3 + 80x^2 - 180x + 144 = 0. \quad (4.6)$$

Thus,

$$(x^4 - 15x^3 + 144)^* = 0. \quad (4.7)$$

Apart from $x = 6$, 3 or 2, $x^* = 4$ yields $(6 - 0 + 4)^* = 0$, so that $x = 4$ and $y = 3/2$. Hence,

$$\{(x, y)\} = \{(6,1),(3,2),(2,3),(4,3/2)\}.$$

3.
$$3x^2 - 5y^2 = 28 \quad (4.8)$$
$$3xy - 4y^2 = 8 \quad (4.9)$$

From (4.8) we see that x and y must have the same parity, and then from (4.9) that they must both be even. Thus $(x^2)^* = 6$ only, but $(y^2)^* = 0$, 4 or 6. Hence $x = \pm4$ or ±6, but $y = 0, \pm2, \pm8, \pm4$ or ±6. Since $y < x$, $y^* \neq 6$ or 8 and $y \neq 0$. Therefore,

$$x = \pm4, \pm6 \text{ and } y = \pm2, \pm4.$$

4.2 Three variables

1.
$$x + 2y + 2z = 11 \quad (4.10)$$
$$2x + y + z = 7 \quad (4.11)$$
$$3x + 4y + z = 14 \quad (4.12)$$

From (4.10) x is odd; from (4.12) z is odd, and so from (4.11) y is even. If we then subtract (4.11) from (4.12) we get

$$x + 3y = 7. \quad (4.13)$$

The x^*, y^* values which satisfy (4.13) are shown in Table 5.

z in Equation	x^*	7	1	5	9	3
	y^*	0	2	4	6	8
(4.10)		2	3			
(4.11)		–	3			
(4.12)	–		3			

Table 5. x^*, y^* values for (4.13)

Thus,

$$x = 1, y = 2, z = 3.$$

Note that if $x = 5r + a$ and $y = 5r + b$, then $a = 1$ and $b = 2$, so that the solution classes are $x \in \overline{1}_5$, $y \in \overline{2}_5$, $z \in \overline{3}_5$.

2.

$$x + 4y + 3z = 17 \quad (4.14)$$

$$3x + 3y + z = 16 \quad (4.15)$$

$$2x + 2y + z = 11 \quad (4.16)$$

From (4.16) z is odd, so from (4.14) x is even, and from (4.15) y is odd. If we then subtract (4.16) from twice (4.14) we get

$$6y + 5z = 23 \quad (4.17)$$

Thus $y^* = 3$ and $y = 3$ fits so solution set is

$$x = 2, y = 3, z = 1.$$

3.

$$2x + 3y + 4z = 20 \quad (4.18)$$

$$3x + 4y + 5z = 26 \quad (4.19)$$

$$3x + 5y + 6z = 31 \quad (4.20)$$

We subtract (4.19) from (4.20) to get

$$y + z = 5 \quad (4.21)$$

From (4.18), y is even, so from (4.21) z is odd and from (4.20) x is odd. Table 6 shows the values for y^*, z^* which are compatible with (4.21).

y^*	0	2	4	6	8
z^*	5	3	1	9	7

Table 6: y^*, z^* values for (4.21)

Hence, $y \neq 6, 8$ (too large) and $y \neq 0, 4$ from Equations (4.18) and (4.20). Thus the solution set is

$$x = 1, y = 2, z = 3.$$

5 Complex roots

1. $$x^4 + x^3 - 6x^2 - 15x - 9 = 0. \quad (5.1)$$

Let $x = 5r + a$. Then, if the constant in (5.1) is to be cancelled when this value of x is substituted, $a = 3$, which yields $r = 0$ and $x = 3$. Equation (5.1) now reduces to

$$x^3 + 4x^2 + 6x + 3 = 0. \quad (5.2)$$

Again let $x = 5r + a$ and substitute into (5.2). Elimination of the constant occurs when $a = -1$, and since $r = 0$ is a root then $x = -1$. Equation (5.2) then reduces to

$$x^2 + 3x + 3 = 0. \quad (5.3)$$

We then use $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ again to get the complex roots, with the complete solution set

$$x = -1, 3, \frac{-3 \pm \sqrt{3}i}{2}$$

Note that $5r - 1 = (5(s-1) + 4) \in \bar{4}_5$.

2. $$x^4 - 3x^3 + 12x - 16 = 0. \quad (5.4)$$

Let $x = 5r + a$. Elimination of the constant yields $a = \pm 2$ which, when substituted into (5.4) both reduce the latter to

$$x^3 - x^2 - 2x + 8 = 0. \quad (5.5)$$

and $r = 0$ is one solution so that $x = \pm 2$.
If $x = 5r - 2$ (to eliminate 8), then (5.5) becomes

$$25r^2 - 35r + 14 = 0. \quad (5.6)$$

so that

$$r = \frac{7 \pm \sqrt{7}i}{10}, \quad (5.7)$$

or $x = 5r - 2 = \frac{3 \pm \sqrt{7}i}{2}$. The solution set is then $\{\pm 2, \frac{3 \pm \sqrt{7}i}{2}\}$.

3. $$x^3 + 1 = 0. \quad (5.8)$$

With $x = 5r - 1$ the constant is eliminated. Substitution into (5.8) yields $r = 0$ and

$$25r^2 - 15r + 3 = 0. \quad (5.9)$$

or

$$r = \frac{3 \pm \sqrt{3}i}{10}, \quad (5.10)$$

so that the solution set is $\{-1, (3 \pm \sqrt{3}i)/2\}$. (Note that $5r - 2 = (5(s-1) + 3) \in \bar{3}_5$ in row $(s-1)$.)

6 Final comments

Students should find the approach outlined here in the examples an interesting alternative to what can sometimes degenerate into a ‘symbol-shoving’ exercise. There is also potential for project work which extends this type of analysis to other types of equations and using other modular rings [6].

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