Notes on Number Theory and Discrete Mathematics
Vol. 18, 2012, No. 3, 35–37

A new elementary proof of the inequality $\varphi(n) > \pi(n)$

Carlo Sanna

Università degli Studi di Torino, Italy
e-mail: `carlo.sanna.dev@gmail.com`

Abstract: In this paper we provide a new elementary proof that the inequality $\varphi(n) > \pi(n)$ holds for all integers $n \geq 91$, an old result of L. Moser. Our proof is based on Bonse's Inequality. This makes it somewhat simpler than Moser's proof, which in turn relies on Bertrand's Postulate.
Keywords: Arithmetic functions, Inequalities.
AMS Classification: 11A25.

1 Introduction

For any positive integer n, we denote by $\pi(n)$ the number of (natural) primes not exceeding n and by $\varphi(n)$ the number of positive integers not greater than n and relatively prime to n. In 1951, L. Moser [4] obtained the solutions of the equation $\varphi(n) = \pi(n)$, by proving that $\varphi(n) > \pi(n)$ holds for all $n \geq 91$. The main tool of Moser's proof is the following lemma, classically referred to as Bertrand's Postulate.

Lemma 1. *$p_{k+1} \leq 2p_k$ for all positive integers k.*

Here and later, p_k stands for the k-th prime number. Elementary proofs of Lemma 1 were given by S. Ramanujan, P. Erdős and L. Moser himself [1, 3, 5]. In this paper, we provide another proof that $\varphi(n) > \pi(n)$ for all integers $n \geq 91$, but in place of Lemma 1 we use Bonse's Inequality [2] in the following form:

Lemma 2. *$p_{k+1} \leq \sqrt{p_1 p_2 \cdots p_k}$ for all integers $k \geq 4$.*

This is motivated by the general desire to find (more) elementary proofs of known results, according to the usual sense attributed to the word "elementary" in the context of number theory. In fact, Lemma 2 is notably weaker than Lemma 1 and we dare to say that the proof of the former is somewhat easier than any up-to-date proof of the latter. Moreover, many proofs of Lemma 1 involve upper bounds on $\pi(n)$ which could be used directly to prove, without much effort, that the inequality $\varphi(n) > \pi(n)$ is eventually true.

2 The inequality $\varphi(n) > \pi(n)$

Let $\mathbb{P}$ be the set of all primes. For any $n \in \mathbb{N}^+$ and $x \geq 0$, we denote by $\omega(n)$ the number of distinct prime factors of n and by $\mathcal{T}_n(x)$ the set of all positive integers not exceeding x and relatively prime with n.

Lemma 3. $|\mathcal{T}_n(x)| \geq \dfrac{x}{\omega(n)+1} - 2^{\omega(n)-1}$ *for any integer* $n \geq 2$ *and real* $x \geq 0$.

Proof. Let $\mu(\cdot)$ be the Möbius function. From the Inclusion-Exclusion Principle it follows that

$$\begin{aligned} |\mathcal{T}_n(x)| &= \sum_{d \,|\, n} \mu(d) \left|\{k \in \mathbb{N}^+ : k \leq x,\, d \mid k\}\right| = \sum_{d \,|\, n} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor \\ &\geq x \sum_{d \,|\, n} \frac{\mu(d)}{d} - \sum_{d \,|\, n,\, \mu(d)=1} 1 = x\,\frac{\varphi(n)}{n} - 2^{\omega(n)-1}. \end{aligned} \tag{1}$$

On the other hand, we have

$$\frac{\varphi(n)}{n} = \prod_{\mathbb{P} \ni p \,|\, n} \frac{p-1}{p} \geq \prod_{i=2}^{\omega(n)+1} \frac{i-1}{i} = \frac{1}{\omega(n)+1}, \tag{2}$$

since the last product in (2) is telescoping. Combining (1) and (2) gives the claim. □

Theorem 1. $\varphi(n) > \pi(n)$ *for any integer* $n \geq 91$.

Proof. Pick an integer $n \geq 2$. We can subdivide $\mathcal{T}_n(n)$ into two disjoint sets: the one consisting of prime numbers less than n and not dividing n; and the one consisting of positive integers less than n, relatively prime with n, but not in $\mathbb{P}$. It follows that

$$\varphi(n) = |\mathcal{T}_n(n)| = \pi(n) - \omega(n) + |\mathcal{T}_n(n) \setminus \mathbb{P}|. \tag{3}$$

Let p be the least prime number such that $p \nmid n$. If $m \in \mathcal{T}_n(n/p) \setminus \{1\}$ then $mp \leq n$, $\gcd(mp, n) = 1$ and $mp \notin \mathbb{P}$, hence $mp \in \mathcal{T}_n(n) \setminus \mathbb{P}$. Therefore, we have

$$|\mathcal{T}_n(n) \setminus \mathbb{P}| \geq |\mathcal{T}_n(n/p)| - 1 \geq \frac{n}{p\,(\omega(n)+1)} - 2^{\omega(n)-1} - 1, \tag{4}$$

as a consequence of Lemma 3. From (3) and (4) it follows

$$\varphi(n) - \pi(n) = |\mathcal{T}_n(n) \setminus \mathbb{P}| - \omega(n) \geq \frac{n}{p(\omega(n)+1)} - 2^{\omega(n)-1} - \omega(n) - 1. \tag{5}$$

Assume at this point that $n \geq 716$. Then, it is uniquely determined an integer $k \geq 4$ such that $n \in [p_k\#, p_{k+1}\# - 1]$, where $p_k\# := p_1 p_2 \cdots p_k$ is the primorial. This implies that $\omega(n) \leq k$ and $p \leq p_{k+1}$. If $k \leq 6$, then computation verifies the inequality

$$\frac{n}{p} \geq \frac{\max(716, p_k\#)}{p_{k+1}} > (2^{k-1} + k + 1)(k+1). \tag{6}$$

Otherwise, Lemma 2 yields $p_{k+1} \le \sqrt{p_k\#}$, whence

$$\begin{aligned} \frac{n}{p} &\ge \frac{p_k\#}{p_{k+1}} \ge \sqrt{p_k\#} \ge \sqrt{p_6\#} \cdot 2^{4(k-6)} > 2^{4(k-6)+7} \ge 9 \cdot 2^{2(k-4)} \\ &= (2^{k-1} + 2^{k-4}) \cdot 2^{k-4} \ge (2^{k-1} + k + 1)(k+1), \end{aligned} \tag{7}$$

since $p_i > 2^4$ for any integer $i \ge 7$ and $k + 1 \le 2^{k-4}$. In both cases, it is found that

$$\frac{n}{p} > (2^{k-1} + k + 1)(k+1) \ge (2^{\omega(n)-1} + \omega(n) + 1)(\omega(n) + 1), \tag{8}$$

which ultimately gives, together with (5), that $\varphi(n) > \pi(n)$. The claim now follows by a direct inspection of the remaining values of n in beetween 90 and 716. This can be sped up by avoiding the computation of many values of $\varphi(n)$ and $\pi(n)$ when (5) already guarantees that $\varphi(n) > \pi(n)$. □

References

[1] Archibald, C., Bertrand's Postulate, *Scripta Mathematica,* Vol. 1, 1945, 109–120.

[2] Bonse, H., Uer eine bekannte Eigenschaft der Zahl 30 und ihre Verallgemeinerung, *Arch. Math. Phys.,* Vol. 12, 1907, 292–295.

[3] Moser, L., A theorem on the distribution of primes, *Amer. Math. Monthly,* Vol. 56, 1949, 624–625.

[4] Moser, L., On the equation $\varphi(n) = \pi(n)$. *Pi Mu Epsilon J.,* 1951, 101–110.

[5] Ramanujan, S., A proof of Bertrand's Postulate, *Journal of the Indian Mathematical Society,* Vol. 11, 1919, 181–182.