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Fibonacci numbers at most one away from a product of factorials

Diego Marques

Department de Mathemática, Universidade de Brasília
Brasília, DF Brazil
e-mail: diego@mat.unb.br

Abstract: Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_0 = 0,\ F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$. In this note, we find all solutions of the Diophantine equation

$$m_1! \cdots m_k! \pm 1 = F_m,$$

where $2 \leq m_1 \leq \cdots \leq m_k$ and $m \geq 3$.
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1 Introduction

Recall that the *factorial* of a positive integer n, denoted by $n!$, is the product of all positive integers less than or equal to n. The well-known *Stirling's formula* (see [1, p. 58]) asserts that $n!$ grows asymptotically as $\sqrt{2\pi n}\,(n/e)^n$ and in fact, it is easy to prove that

$$n! > \left(\frac{n}{e}\right)^n, \text{ for all } n \geq 1. \tag{1}$$

In the past years, several authors have considered Diophantine equations involving factorial numbers. For instance, Erdös and Selfridge [8] proved that $n!$ is a perfect power, only when $n = 1$. However, the most famous among such equations was posed by Brocard [5], in 1876, and independently by Ramanujan [17], [18, p. 327], in 1913. The Diophantine equation

$$n! + 1 = m^2 \tag{2}$$

is then known as *Brocard-Ramanujan Diophantine equation*.

It is a simple matter to find the three known solutions, namely when $n = 4, 5$ and 7. Recently, Berndt and Galway [2] did not find further solutions up to $n = 10^9$. The best contribution to the problem is due to Overholt [16], who showed that the equation (2) has only finitely many solutions if we assume a weak version of the abc Conjecture[1]. However, the Brocard-Ramanujan equation is still an open problem.

Let $(F_n)_{n\geq 0}$ be the *Fibonacci sequence* given by $F_0 = 0,\ F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$. There are several interesting problems related to Fibonacci numbers. For instance, the problem of finding the perfect powers in the Fibonacci sequence was a classical problem that attracted much attention during the past 40 years. In 2003, Bugeaud et al [6, Theorem 1] confirmed the expectation: the only perfect powers in that sequence are $0, 1, 8$ and 144. A generalization of this result can be found in [15]. Consult the historical section of [6] and its very extensive annotated bibliography for additional references and history. We still point out that Sun [19] has recently conjectured that every integer $n > 4$ can be written as the sum of an odd prime and two positive Fibonacci numbers.

A number of mathematicians have been interested in Diophantine equations that involve both factorial and Fibonacci numbers. For example, in [9] it is shown that if k is fixed, then there are only finitely many positive integers n such that

$$F_n = m_1! + m_2! + \cdots + m_k!$$

holds for some positive integers $m_1, \ldots, m_k$ and all the solutions for the case $k \leq 2$ have been determined. After, the case $k = 3$ was also solved, see [4]. In a very recent paper, Luca and Siksek [11] found all factorials expressible as the sum of at least three Fibonacci numbers.

In 1999, Luca [10], proved that F_n is a product of factorials only when $n = 1, 2, 3, 6, 12$. Also, the largest product of distinct Fibonacci numbers which is a product of factorials is

$$F_1F_2F_3F_4F_5F_6F_8F_{10}F_{12} = 11!,$$

see [12].

In this note, we find Fibonacci numbers whose difference to a product of factorials is ± 1. Our result is the following

Theorem 1. *The only solutions of the Diophantine equation*

$$m_1! \cdots m_k! \pm 1 = F_m, \tag{3}$$

where $2 \leq m_1 \leq \cdots \leq m_k$ *and* $m \geq 3$, *are* $m = 4, 5$ *and* $m = 4, 5, 7$ *in the (-) and (+) cases, respectively. Explicitly, we have* $2! + 1 = (2!)^2 - 1 = F_4,\ (2!)^2 + 1 = 3! - 1 = F_5$ *and* $2!3! + 1 = F_7$.

We organize this paper as follows. In Section 2, we will recall some useful properties such as the Binet's formulae, bounds on Fibonacci and Lucas numbers and the method of factorization

[1] This conditional result gained more importance, since very recently, S. Mochizuki wrote a paper with a serious claim to a proof of the abc Conjecture.

for $F_n \pm 1$ that we will use to prove Theorem 1. The third section is devoted to the proof of Theorem 1. In the last section, we shall search for pairs (n, m), where m is a Fibonacci number, satisfying the Brocard-Ramanujan equation.

2 Auxiliary results

Before proceeding further, some considerations will be needed for the convenience of the reader.

The problem of the existence of infinitely many prime numbers in the Fibonacci sequence remains open, however several results on the prime factors of a Fibonacci number are known. For instance, a *primitive divisor* p of F_n is a prime factor of F_n which does not divide $\prod_{j=1}^{n-1} F_j$. It is known that a primitive divisor p of F_n exists whenever $n \geq 13$ and moreover $p \equiv 1 \pmod{n}$. The above statement is usually referred to the *Primitive Divisor Theorem* (see [3] for the most general version).

We cannot go very far in the lore of Fibonacci numbers without encountering the sequence of *Lucas numbers* $(L_n)_{n\geq 0}$ which follows the same recursive pattern as the Fibonacci numbers, but with initial values $L_0 = 2$ and $L_1 = 1$.

By the Binet's formulae, we have

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^n, \text{ for all } n \geq 1,$$

where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2 = (-\alpha)^{-1}$. Therefore, for all $n \geq 1$

$$\alpha^{n-2} < F_n < \alpha^{n-1}. \tag{4}$$

Among the several pretty identities involving Fibonacci and Lucas numbers, we cite that $L_n = F_{n-1} + F_{n+1}$, for $n \geq 1$, which can be easily proved by induction. This formula allows to deduce the following estimate

$$L_n = F_{n-1} + F_{n+1} < \alpha^{n-2} + \alpha^n < 3.7\alpha^{n-2}. \tag{5}$$

We may note that the Fibonacci and Lucas sequences can be extrapolated backwards using $F_n = F_{n+2} - F_{n+1}$ and $L_n = L_{n+2} - L_{n+1}$. Thus, for example, $F_{-1} = 1, F_{-2} = -1$, and so on. Since that the Binet's formulae remain valid for Fibonacci and Lucas numbers with negative indices, one can deduce the following result (which we shall prove for the sake of completeness)

Lemma 1. *For any integers a, b, we have*

$$F_a L_b = F_{a+b} + (-1)^b F_{a-b}.$$

Proof. The identity $\alpha = (-\beta)^{-1}$ leads to

$$F_a L_b = \frac{\alpha^a - \beta^a}{\alpha - \beta}(\alpha^b + \beta^b) = F_{a+b} + \frac{\alpha^a\beta^b - \beta^a\alpha^b}{\alpha - \beta} = F_{a+b} + (-1)^b F_{a-b}.$$

□

Lemma 1 gives immediately the following factorizations for $F_n \pm 1$, depending on the class of n modulo 4:

$$\begin{array}{rcl} F_{4\ell}+1 = F_{2\ell-1}L_{2\ell+1} & ; & F_{4\ell}-1 = F_{2\ell+1}L_{2\ell-1} \\ F_{4\ell+1}+1 = F_{2\ell+1}L_{2\ell} & ; & F_{4\ell+1}-1 = F_{2\ell}L_{2\ell+1} \\ F_{4\ell+2}+1 = F_{2\ell+2}L_{2\ell} & ; & F_{4\ell+2}-1 = F_{2\ell}L_{2\ell+2} \\ F_{4\ell+3}+1 = F_{2\ell+1}L_{2\ell+2} & ; & F_{4\ell+3}-1 = F_{2\ell+2}L_{2\ell+1} \end{array} \tag{6}$$

The next lemma plays an important role in the proof of Theorem 1

Lemma 2. *We have that*

$$\left(\frac{m-4}{2e}\right)^{\frac{m-4}{2}} - 3.7\alpha^{m-1} > \alpha^{2m-13},$$

for all $m \geq 10$.

Proof. For $m \geq 10$, we have $(m-4)/2e > \alpha^4$ and $\alpha^{m-7} - \alpha^{m-12} > 3.7$. Then

$$\left(\frac{m-4}{2e}\right)^{\frac{m-4}{2}} - 3.7\alpha^{m-1} > \alpha^{2m-8} - 3.7\alpha^{m-1} = \alpha^{m-1}(\alpha^{m-7} - 3.7) > \alpha^{2m-13},$$

for $m \geq 10$, which completes the proof.

□

Now, we are ready to deal with the proof of theorem.

3 The proof of Theorem

The equation (3) can be rewritten as $m_1! \cdots m_k! = F_m \mp 1$. By the relations in (6), we have that $F_m \mp 1 = F_a L_b$, where $2a, 2b \in \{m \pm 1, m \pm 2\}$. Therefore, our equation becomes

$$m_1! \cdots m_k! = F_a L_b \tag{7}$$

Now, the estimates in (4), (5) and the identity (7) yield

$$m_1! \cdots m_k! \leq 3.7\alpha^{\frac{m+2}{2}-1}\alpha^{\frac{m+2}{2}-2} = 3.7\alpha^{m-1}. \tag{8}$$

On the other hand, if $m > 26$, then $a \geq (m-2)/2 > 12$ and the Primitive Divisor Theorem implies in the existence of a primitive divisor p of F_a which in particular satisfied $p \equiv 1 \pmod a$. Thus $m_k \geq p \geq a-1 \geq (m-4)/2$ and the estimate (1) yields $m_k! \geq ((m-4)/2e)^{(m-4)/2}$. Therefore, we use (8) to get

$$3.7\alpha^{m-1} \geq m_1! \cdots m_k! \geq m_k! \geq ((m-4)/2e)^{(m-4)/2} > 3.7\alpha^{m-1},$$

for $m \geq 10$, by Lemma 2, which gives an absurd. Hence $3 \leq m \leq 26$ and a straight calculation gives that the only solutions of (3) are those in the statement of the theorem. However, for convenience of the reader we shall describe in a few words how these calculations can be performed. First, note that $m_k! \leq F_{26} \mp 1 \leq 121394$ which implies $m_k \leq 8$. Now, we need an upper bound on k. For that, if $m_j \in \{2, 3, 5\}$, for $1 \leq j \leq k$, in the equation (3), then $F_m \mp 1 = (2!)^a(3!)^b(5!)^c = 2^{a+b+3c}3^{b+c}5^c$. Thus, if $c > 0$, then $40 \mid F_m \mp 1$ which does not happen in the obtained range and so $c = 0$. If $b > 0$, then $(F_m \pm 1)/6 \in \{2^s \cdot 3^\ell : s, \ell \in \mathbb{N}\}$ which implies $m = 7, 10$ and $m = 5, 11$ in the (-) and (+) case, respectively. Since $a + b \geq b$, the only possibilities are $m = 7$ ($F_7 = 2!3! + 1$) and $m = 5$ ($F_5 = 3! - 1$). When $b = 0$, the number $F_m \mp 1$ must be a power of 2 which happens only when $m = 4$ ($F_4 = (2!)^2 - 1 = 2! + 1$) and $m = 5$ ($F_5 = (2!)^2 + 1$). Thus, the other possibilities happens when $m_k > 5$ and so

$$5040 \cdot 2^{k-1} \leq m_1! \cdots m_k! \leq 121394,$$

which implies $k \leq 5$. We now prepare a simple Mathematica routine and one needs a few seconds to show that the difference

$$m_1!m_2!m_3!m_4!m_5! \pm 1 - F_m$$

is never zero in the range $1 \leq m_1 \leq m_2 \leq m_3 \leq m_4 \leq 8$, $7 \leq m_5 \leq 8$ and $3 \leq m \leq 26$. This completes the proof. □

4 Fibonacci numbers in the Brocard-Ramanujam equation

We point out that the idea of writing $F_n \pm 1$ as a product of a Fibonacci and a Lucas numbers has been used for attacking some Diophantine equations involving Fibonacci numbers. For instance, the equation $F_n \pm 1 = y^\ell$ with integer y and $\ell \geq 2$ have been solved in [7]. In a recent paper, the author [13, Theorem 1.1] proved that $F_1 \cdots F_n + 1 = F_m^2$ has no solution in positive integers m, n and after he proved that the equation $F_1 \cdots F_n + 1 = F_m^t$ has only finitely many solutions for each t previously fixed, see [14].

As another application of this method, we shall prove the following result

Theorem 2. *If (n, m) is a solution of the Brocard-Ramanujan equation* (2)*, where m is a Fibonacci number, then $(n, m) = (4, F_5)$.*

Proof. If $n! + 1 = m^2$ and $m = F_k$, then $n! = F_k^2 - 1 = (F_k - 1)(F_k + 1) = F_a L_b F_c L_b$, where $2a, 2b, 2c, 2d \in \{k \pm 1, k \pm 2\}$. By the Berndt and Galway calculations, see [2], we can consider $F_k = m > 10^9$ and so $k \geq 45$. Therefore, the remainder of the proof proceeds along the same lines as the proof of Theorem 1. □

We finish by pointing out that the following general statement can be proved similarly to the proofs of our results: let a be an integer number and let $(C_n)_{n \geq 1}$ be a sequence given by $C_1 = C_2 = 1$ and $C_{n+2} = aC_{n+1} + C_n$. Then, there exists an effectively computable constant K such that if

$$m_1! \cdots m_k! \pm 1 = C_m \text{ and } n! + 1 = C_\ell^2,$$

holds for some positive integers $n, m_1, \ldots, m_k$, then $m, \ell \leq K$. The constant K depends only on the parameters of C_n.

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