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A modification of an elementary numerical inequality

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Abstract: It is proved that for every real numbers $a_0, a_1, ..., a_n$ $(n \geq 1)$:

$$\sum_{k=0}^{n-1} a_{k+1}.(a_k - a_{k+1}) \leq \frac{1}{2.n}.\sum_{k=0}^{n-1} a_k^2.$$

This is a modification of an inequality, previously introduced by the author.
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In memory of Prof. Antal Bege

1 Introduction

Let real numbers $a_0, a_1, ..., a_n$ $(n \geq 1)$ be given, where $a_0 = 1$. In [1] it is proved the validity of the following inequality:

$$\sum_{k=0}^{n-1} a_{k+1}.(a_k - a_{k+1}) \leq \frac{n}{2.(n+1)}.$$

Obviously, this inequality is a generalization of the well-known inequality

$$a.(1-a) \leq \frac{1}{4}.$$

A modification of the above inequality has been discussed by L. Beran and E. Novakova in [2] in the form

$$\sum_{k=0}^{n-1} a_{k+1}.(a_k - a_{k+1}) \leq \frac{na_0^2}{2(n+1)}$$

for given real numbers $a_0, a_1, ..., a_n$ $(n \geq 1)$.

A modification of the last inequality is introduced and proved in another way by I. Coope and P. Renaud in [3] in the form

$$\sum_{k=1}^{n-1} a_{k+1}.(a_k - a_{k+1}) \leq \frac{(n-1)a_1^2}{2.n}$$

or

$$a_2.(a_1 - a_2) + a_3.(a_2 - a_3) + ... + a_n.(a_{n-1} - a_n) \leq \frac{(n-1)a_1^2}{2.n} \tag{2}$$

for given real numbers $a_1, ..., a_n$ $(n \geq 2)$.

2 Main results

Now, we shall construct a new form of the initial inequality, modifying (2) to the form

$$a_1.(a_0 - a_1) + a_2.(a_1 - a_2) + a_3.(a_2 - a_3) + ... + a_0.(a_{n-1} - a_0) \leq \frac{(n-1)a_1^2}{2.n} + a_1.(a_0 - a_1), \tag{3}$$

where $a_0 = a_n$.

Let us define

$$A \equiv a_1.(a_0 - a_1) + a_2.(a_1 - a_2) + a_3.(a_2 - a_3) + ... + a_0.(a_{n-1} - a_0).$$

Therefore, (3) has the form

$$A \leq \frac{(n-1)a_1^2}{2.n} + a_1.(a_0 - a_1). \tag{4'}$$

Now, let us put in (3) a_{i+1} instead of a_i for $i = 1, 2, ..., n-1$. Thus we obtain

$$a_2.(a_1 - a_2) + a_3.(a_2 - a_3) + a_4.(a_3 - a_4) + ... + a_1.(a_0 - a_1) \leq \frac{(n-1)a_2^2}{2.n} + a_2.(a_1 - a_2), \tag{5}$$

and hence

$$A \leq \frac{(n-1)a_2^2}{2.n} + a_2.(a_1 - a_2). \tag{4''}$$

Repeating this substitution in (5), we obtain

$$a_3.(a_2 - a_3) + a_4.(a_3 - a_4) + a_5.(a_4 - a_5) + ... + a_{n-1}.(a_0 - a_{n-1}) \leq \frac{(n-1)a_2^2}{2.n} + a_3.(a_2 - a_3),$$

i.e.,

$$A \leq \frac{(n-1)a_3^2}{2.n} + a_3.(a_2 - a_3) \tag{4'''}$$

etc. Summarizing the inequalities $(4')$, $(4'')$, $(4''')$, etc., that are n in number, we obtain inequality

$$n.A \leq \frac{(n-1)}{2.n}.\sum_{k=0}^{n-1} a_k^2 + A.$$

Therefore,

$$A \leq \frac{1}{2.n}.\sum_{k=0}^{n-1} a_k^2.$$

Hence, it is valid the following

Theorem: For every real numbers $a_0, a_1, ..., a_n$ $(n \geq 1)$:

$$\sum_{k=0}^{n-1} a_{k+1}.(a_k - a_{k+1}) \leq \frac{1}{2.n}.\sum_{k=0}^{n-1} a_k^2.$$

It is a new form of the initial inequality.

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References

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