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Short remarks on Jacobsthal numbers

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Abstract: Some new generalization of the Jacobsthal numbers are introduced and properties of the new number are studied.
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The n-th Jacobsthal number ($n \geq 0$) is defined by

$$J_n = \frac{2^n - (-1)^n}{3} \tag{1}$$

(see, e.g., [1]). In [2], it is generalized to the form

$$J_n^s = \frac{s^n - (-1)^n}{s+1}, \tag{2}$$

where $n \geq 0$ is a natural number and $s \geq 0$ is a real number.

The presumption of this generalization is that number 2 is changed with s and therefore, 3 must be changed with $s + 1$.

Here another generalization is introduced, interpreting 3 not as the next number after 2, but as $2^2 - 1$, i.e., changing it by $s^2 - 1$. In a result, the following new numbers are obtained

$$Y_n^s = \frac{s^n - (-1)^n}{s^2 - 1}, \tag{3}$$

where $s \neq 1$ is a real number.

Obviously, when $s = 2$ we obtain the standard Jacobsthal numbers.

In the case $s = 0$, we obtain

$$Y_n^0 = (-1)^n.$$

The first six members of the sequence $\{Y_n^s\}$ with respect to n are

0	1	2	3	4	5
0	$\frac{1}{s-1}$	1	$s + \frac{1}{s-1}$	$s^2 + s + \frac{1}{s-1}$	$s^3 + s + \frac{1}{s-1}$

It can be directly seen from (2) and (3) that for the real number $s \neq 1$

$$Y_n^s = \frac{1}{s-1}.J_n^s.$$

Theorem 1. For every natural number $n \geq 0$ and real number $s \neq 1$:

$$Y_{n+2}^s = Y_n^s + s^n,$$

$$Y_{n+1}^s = s.Y_n^s + \frac{(-1)^n}{s-1}.$$

Proof. It can be directly checked that for each $n \geq 0$:

$$Y_{n+2}^s - Y_n^s = \frac{1}{s^2-1}.(s^{n+2} - (-1)^{n+2} - s^n + (-1)^n) = s^n.$$

$$Y_{n+1}^s - s.Y_n^s = \frac{1}{s^2-1}.(s^{n+1} - (-1)^{n+1} - s^{n+1} + (-1)^n.s) = \frac{(-1)^n}{s-1}.$$

Two next steps of modification of the Jacobsthal numbers have the following forms.

First, we can mention that 2 and 3 are the first two prime numbers and therefore, the Jacobstal numbers can obtain the form

$$JP_n^s = \frac{p_s^n - (-1)^n}{p_{s+1}}, \tag{4}$$

where p_i is the i-th prime number ($p_0 = 2, p_1 = 3, ...$).

Second, we can mention that 2 and 3 are two sequential Fibonacci numbers and, therefore, the Jacobstal numbers can obtain the form

$$JF_n^s = \frac{f_s^n - (-1)^n}{f_{s+1}}, \tag{5}$$

where f_i is the i-th Fibonacci number ($f_0 = 0, f_1 = 1, f_2 = 1, f_3 = 2, f_4 = 3, ...$).

For the latest numbers we see that the following assertion is valid.

Theorem 2. For every natural number $n \geq 0$ and real number $s \neq 1$:

$$JF_{n+1}^s = JF_n^s + s^n + \frac{f_s - 1}{f_{s+1}}.f_s^n.$$

An **open problem** is to study the properties of numbers JP_n^s and JF_n^s.

References

[1] Ribenboim, P. *The Theory of Classical Variations,* Springer, New York, 1999.

[2] Atanassov K., Remark on Jacobsthal numbers, Part 2. *Notes on Number Theory and Discrete Mathematics*, Vol. 17, 2011, No. 2, 37–39. http://nntdm.net/papers/nntdm-17/NNTDM-17-2-37-39.pdf