Notes on Number Theory and Discrete Mathematics
Vol. 18, 2012, No. 2, 56–57

Quotients of primes in arithmetic progressions

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Abstract: We prove an open problem of Hobby and Silberger on quotients of primes in arithmetic progressions.

Keywords: Arithmetic progression, Prime number.

AMS Classification: 11A25, 11A41.

Let a and b be positive coprime integers and denote by $\mathbf{D}(a, b)$ the set of prime numbers which are congruent to b modulo a. For a given set $S \subseteq \mathbf{N}$ write $\mathbf{F}(S)$ for the set of all quotients of elements of S.

In 1993, Hobby and Silberger [1] proved that if $\mathbb{P}$ is the set of all prime numbers, then $\mathbf{F}(\mathbb{P})$ is dense in $\mathbf{R}^+ := [0, \infty)$. As an open problem they asked for the generalization of this result to arithmetic progressions; that is, decide whether $\mathbf{F}(\mathbf{D}(a, b))$ is dense in $\mathbf{R}^+$. Two years later, Starni [2] claimed to answer Hobby and Silberger's question in the affirmative, though it seems that their proof has a flaw. Indeed, if we write $p_{a,b}(n)$ for the n-th prime in $\mathbf{D}(a, b)$, Starni claimed that $p_{a,b}(n) \sim n \log n$ which is false as we shall prove in the following lemma.

Lemma. *If $p_{a,b}(n)$ is as defined above, then $p_{a,b}(n) \sim \varphi(a)\, n \log n$, where $\varphi(q)$ is Euler's totient function.*

Proof. Denote by $\pi(x; a, b)$ the number of primes up to x that are congruent to b modulo a. The prime number theorem for arithmetic progressions thus implies that

$$\lim_{n\to\infty} \frac{\varphi(a) n \log p_{a,b}(n)}{p_{a,b}(n)} = 1.$$

Taking logarithms and dividing by $\log p_{a,b}(n)$ we obtain

$$\lim_{n\to\infty} \frac{\log n}{\log p_{a,b}(n)} = 1.$$

Multiplying the two above limits together proves the lemma. □

Note that Starni's claim with this lemma, Starni's prove goes through nicely. To make this paper self contained and a bit more interesting, we use the above lemma to give a simpler proof of the density of **F**(**D**(a, b)) is dense in $\mathbf{R}^+$.

Theorem. *We have* **F**(**D**(a, b)) *is dense in* $\mathbf{R}^+$.

Proof. Let $[y]$ denote the integer part of the positive real number y. By the above lemma we have for a given real number $x \in \mathbb{R}^+$ that

$$\lim_{n\to\infty} \frac{p_{a,b}([xn])}{p_{a,b}(n)} = x. \qquad \square$$

We note that this type of argument was suggested by M. Mendès France in his review of [1].

References

[1] Hobby, D., D. M. Silbeger, Quotients of primes, *Amer. Math. Monthly*, Vol. 100, 1993, No. 1, 50–52.

[2] Starni, P., Answers to two questions concerning quotients of primes, *Amer. Math. Monthly*, Vol. 102, 1995, No. 4, 347–349.