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Row-wise representation of arbitrary rhotrix

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Abstract: This paper identifies some various methods of representing an arbitrary rhotrix. One of the methods - the *row-wise* method - has been chosen as it is observed to be flexible in analysing rhotrices for mathematical enrichment. A relationship between the location of the heart of a rhotrix and the dimension of the rhotrix and also a relationship between the location of the heart of a rhotrix and the order of the principal matrix of the rhotrix have been determined. The flexibility of the representation has paved way for two formulae, one for row-column multiplication of arbitrary rhotrices and the other for heart-oriented multiplication of arbitrary rhotrices. Some examples have also been given as a way of demonstrating the application of the proposed formulae. Finally, the paper introduces the concepts of subrhotrix and submatrix of a rhotrix which can be exploited for further study of various algebraic properties of rhotrices.

Keywords: rhotrix, principal matrix, complementary matrix, inscribed matrix, row-wise representation, row-column multiplication, heart-oriented multiplication, subrhotrix, submatrix.
AMS Classification: N/A

1 Introduction

Ajibade [1] introduced the concept of *rhotrix* owing to some ideas he concieved of *matrix-tertions* and *matrix-noitrets* [2]. He identified a rhotrix as an object which is in some ways between a (2×2) – dimensional matrix and a (3×3) – dimensional matrix. He stated that the name 'rhotrix' was coined from the rhomboid shape of the object. He further noted that an extension is possible in various ways. One way is for the object to take the shape of a *kite*, and is called a *skew rhotrix* in this paper.

Row-column multiplication of rhotrices was suggested by Sani [5], as an alternative to the multiplication in [1]. This multiplication was generalized in [6]. We prove a theorem which establishes the formula describing this multiplication. By the use of the formula, we have an alternative to performing row-column multiplication one matrix at a time [6].

2 Index-based representation of an arbitrary rhotrix

Unlike in an arbitrary matrix where the indices of an entry can uniquely be identified by the rows and columns of the matrix, there are various ways of representing the entries of an arbitrary rhotrix. Usually, an author would employ a particular method to suit the usage of the object.

One way of representing an arbitrary rhotrix is by the use of a single index for each entry as in the following:

$$\left\langle \begin{array}{ccccc} & & a_1 & & \\ & a_6 & a_4 & a_2 & \\ a_{11} & a_9 & a_7 & a_5 & a_3 \\ & a_{12} & a_{10} & a_8 & \\ & & a_{13} & & \end{array} \right\rangle.$$

This method is called *single-index* method of arbitrary rhotrix representation. The indices can also be allowed to run horizontally from left to right.

Another way is to use two indices, the first indicating the row in which the entry lies, and the second indicating the column in which the entry lies as in the following rhotrix. See for the definitions of a row and a column of a rhotrix in Section 4.

$$\left\langle \begin{array}{ccccc} & & a_{11} & & \\ & a_{31} & a_{22} & a_{13} & \\ a_{51} & a_{42} & a_{33} & a_{24} & a_{15} \\ & a_{53} & a_{44} & a_{35} & \\ & & a_{55} & & \end{array} \right\rangle.$$

This method is called *row-column* method of representing the entries of an arbitrary rhotrix since each entry can be identified by its row and column. In this method the location (or position) of a row in a rhotrix is identified by the first index in its entries while that of a column is identified by the second index in its entries. Thus, the first row has 1 as the first index in its entries while the third column has 3 as the second index in its entries and so on. It is important not to confuse this with row-column multiplication of rhotrices.

A third method also uses two indices for each entry, where the first index indicates the row in which the entry lies. However, the second index does not indicate the column in which the entry lies in the rhotrix. This type of rhotrix representation can be seen in the following rhotrix:

$$\left\langle \begin{array}{ccccc} & & a_{11} & & \\ & a_{31} & a_{21} & a_{12} & \\ a_{51} & a_{41} & a_{32} & a_{22} & a_{13} \\ & a_{52} & a_{42} & a_{33} & \\ & & a_{53} & & \end{array} \right\rangle. \tag{1}$$

It is this method of arbitrary rhotrix representation that is our focus in this paper, and is termed *row-wise* method of arbitrary rhotrix representation.

From the above, an arbitrary rhotrix can be represented owing to its need at a particular point in time. It is also pertinent to emphasize that any form of representation is as far as its aim can be achieved. Also, the flexibility in representation can be seen to explain the superiority of rhotrices over matrices for mathematical enrichment. However, this is not without the fact that proper handling is neccessary as in other mathematical tools. By so doing, rhotrix can go a long way to showcasing itself as a powerful tool in mathematics and science in general.

3 Compelling circumstances for row-wise method of arbitrary rhotrix representation

The use of two different indexed letters to represent the entries of an arbitrary rhotrix as in

$$\langle a_{ij}, c_{kl}\rangle = \left\langle \begin{array}{ccccc} & & a_{11} & & \\ & a_{21} & c_{11} & a_{12} & \\ a_{31} & c_{21} & a_{22} & c_{12} & a_{13} \\ & a_{32} & c_{22} & a_{23} & \\ & & a_{33} & & \end{array} \right\rangle$$

seems to have first appeared in [6]. One reason for the representation is probably because the first and second indices of an entry in the rhotrix and the position of the row and column of either of the embedded matrices

$$(a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \text{ and } (c_{kl}) = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

in which the entry lies in the matrix correspond, respectively. However, from the point of view of the row-wise representation, we devise a rule (see Section 5) which transforms these matrices from the form they inherited from row-wise arbitrary rhotrix representation into standard form of arbitrary matrix representation. The matrices (a_{ij}) and (c_{kl}) are, in this paper, called the *principal* and *complementary* matrices of the rhotrix $\langle a_{ij}\ , c_{kl}\rangle$, respectively. Some ‘informal’ definitions of these terms are given in the next section.

Moreover, if one is to represent the *skew rhotrix* in (2), one would appreciate the use of the row-wise method.

This is because it is not possible to embed the principal and complementary matrices in the skew rhotrix in such a way that each of all the entries in the skew rhotrix is an entry in either of the two matrices without a rearrangement of entries. Note that a skew rhotrix is to rhotrices just as a rectangular matrix is to matrices.

$$
\left\langle \begin{array}{ccccc}
 & & a_{11} & & \\
 & a_{31} & a_{21} & a_{12} & \\
a_{51} & a_{41} & a_{32} & a_{22} & a_{13} \\
a_{61} & a_{52} & a_{42} & a_{33} & a_{23} \\
 & a_{62} & a_{53} & a_{43} & \\
 & a_{71} & a_{63} & a_{54} & \\
 & a_{81} & a_{72} & a_{64} & \\
 & & a_{82} & & \\
 & & a_{91} & &
\end{array} \right\rangle \tag{2}
$$

Also, a rhotrix in row-wise form can conveniently be represented by a multiset permutation. For instance, (1) is expressed as

$$V = [[a_{11}, a_{12}, a_{13}], [a_{21}, a_{22}], [a_{31}, a_{32}, a_{33}], [a_{41}, a_{42}], [a_{51}, a_{52}, a_{53}]].$$

Note that each member of V is called an element-multiset (or an element) of V. Note also that V is a permutation (an ordered sequence of elements with repetitions allowed [8]), and so are its element-multisets, thus the use of square brackets. Therefore, a re-arrangement of the element-multisets of V or of the elements of any element-multiset of V is not necessarily equal to V. The second index j can be seen to indicate the position of each element in an element-multiset, just as the first index indicates the position of the element-multisets of V. The skew rhotrix in (2) can similarly be represented in the form of a multiset permutation by

$$[[a_{11}, a_{12}, a_{13}], [a_{21}, a_{22}, a_{23}], [a_{31}, a_{32}, a_{33}], [a_{41}, a_{42}, a_{43}], [a_{51}, a_{52}, a_{53}, a_{54}], \ [a_{61}, a_{62}, a_{63}, a_{64}],$$

$$[a_{71}, a_{72}], [a_{81}, a_{82}], [a_{91}]].$$

It is also relevant to note that the representation of a rhotrix as a multiset permutation proves useful in computational parlance [8].

4 Some preliminary concepts

Definition i: The axes of a rhotrix A *vertical axis* of a rhotrix is an array of entries running from the top to the bottom of the rhotrix while a *horizontal axis* of a rhotrix is an array of entries running from the left to the right hand side of the rhotrix. This is the same as the diagonals in Ajibade [1]. Every rhotrix has a *major vertical* axis and a *major horizontal* axis. For example, consider the rhotrix

$$R_5 = \left\langle \begin{array}{ccccc} & & a & & \\ & f & d & b & \\ k & i & g & e & c \\ & l & j & h & \\ & & m & & \end{array} \right\rangle$$

The vertical axes of R_5 include:

k , $\begin{array}{c} f \\ i \\ l \end{array}$ and $\begin{array}{c} a \\ d \\ g \\ j \\ m \end{array}$ while the horizontal axes include m, $f \quad d \quad b$ and $k \quad i \quad g \quad e \quad c$; the major axes being the last of the two lists.

Definition ii: The dimension of a rhotrix is the number of entries in a major axis of the rhotrix. The dimension of a rhotrix is sometimes attached as a subscript to the variable used to indicate the rhotrix. For instance a rhotrix of the fifth dimension may be denoted as R_5.

Definition iii: The heart of a rhotrix is the entry located at the perpendicular intersection of the two major axes of a rhotrix. In other words, it is the entry at the center of the rhotrix. That is, given a rhotrix

$$R = \{ \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle : a, b, c, d, e \in \Re \}$$

of dimension 3, the heart $\mathrm{h}(R)$ of R is identified by the entry c and we write

$$R = \left\langle \begin{array}{ccc} & a & \\ b & \mathrm{h}(R) & d \\ & e & \end{array} \right\rangle .$$

Being of dimension 3 we identify the rhotrix by R_3 ([1], for details).

Definition iv: The rows and columns of a rhotrix We follow the definitions of a row and a column of a 3-dimensional rhotrix [5]. A *row* of a rhotrix is an array of entries running from the top-left to the right-bottom side of the rhotrix. A *column* of a rhotrix is an array of entries running from the top-right to the left-bottom side of the rhotrix.

For example, consider the 5-dimensional rhotrix R_5 in Definition i above. The first row is $\begin{array}{ccc} a & & \\ & b & \\ & & c \end{array}$, the second row is $\begin{array}{cc} d & \\ & e \end{array}$ and the third row is $\begin{array}{ccc} f & & \\ & g & \\ & & h \end{array}$. On the other hand, the first column is $\begin{array}{ccc} & & a \\ & f & \\ k & & \end{array}$, the second column is $\begin{array}{cc} & d \\ i & \end{array}$ while the fifth column is $\begin{array}{ccc} & & c \\ & h & \\ m & & \end{array}$.

Definition v: Heart-oriented multiplication of rhotrices Ajibade [1] defined rhotrix multiplication '$\circ$' as follows:
For any two rhotrices

$$R = \left\langle \begin{array}{ccc} & a & \\ b & \mathrm{h}(R) & d \\ & e & \end{array} \right\rangle \text{ and } Q = \left\langle \begin{array}{ccc} & f & \\ g & \mathrm{h}(Q) & j \\ & k & \end{array} \right\rangle,$$

$$R \circ Q = \left\langle \begin{array}{ccc} & a\mathrm{h}(Q) + \mathrm{h}(R)f & \\ b\mathrm{h}(Q) + \mathrm{h}(R)g & \mathrm{h}(R)\mathrm{h}(Q) & d\mathrm{h}(Q) + \mathrm{h}(R)j \\ & e\mathrm{h}(Q) + \mathrm{h}(R)k & \end{array} \right\rangle.$$

This form of rhotrix multiplication is called *heart-oriented* multiplication of rhotrices [3]. Note that $\mathrm{h}(R)$ and $\mathrm{h}(Q)$ are the hearts of R and Q respectively.

Definition vi: Row-column multiplication of rhotrices Sani [5, 6] defined *row-column multiplication* of 3-dimensional rhotrices as follows:
Let

$$A = \left\langle \begin{array}{ccc} & a & \\ d & c & b \\ & e & \end{array} \right\rangle, B = \left\langle \begin{array}{ccc} & f & \\ i & h & g \\ & j & \end{array} \right\rangle$$

be two rhotrices of the third dimension. The row-column product '$\circ$' of *A* and *B* is expressed as

$$A \circ B = \left\langle \begin{array}{ccc} & af + bi & \\ df + ei & ch & ag + bj \\ & dg + ej & \end{array} \right\rangle.$$

Definition vii: The boundary entries and boundary of a rhotrix or matrix An entry at the first row, last row, first column or last column of a rhotrix/ matrix is called a *boundary entry* of the rhotrix/matrix. All the entries in a single row or a single column matrix are boundary entries. Moreover, there is only one boundary entry in a rhotrix/matrix with only a single entry. Furthermore, the multiset [8] of all the boundary entries in a rhotrix/matrix is called the *boundary* of the rhotrix/matrix. For instance, consider the rhotrix

$$R_5 = \left\langle \begin{array}{ccccc} & & 5 & & \\ & 6 & 2 & 3 & \\ 5 & 0 & 1 & 1 & 5 \\ & 4 & 3 & 4 & \\ & & 9 & & \end{array} \right\rangle \text{and the matrix } M = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 0 & 1 \\ 2 & 9 & 1 \end{pmatrix}.$$

The boundary of R_5 is $\{3, 4, 4, 5, 5, 5, 6, 9\}$ while the boundary of M is $\{1, 1, 1, 1, 2, 2, 3, 9\}$. The boundary entries are the elements of the boundaries.

Definition viii: The vertex of a rhotrix or matrix A *vertex* of a rhotrix/matrix is an entry at any of the four corners of the rhotrix/matrix. For instance, using the rhotrix and matrix in Definition vii above, the vertices of R_5 are 5, 5, 5 and 9 while the vertices of M are 3, 1, 1 and 2.

Definition ix: The principal matrix of a rhotrix is a 'tilted' matrix with the largest dimension in the rhotrix. It is called *principal* because it shares the same boundary with the host rhotrix. By way of illustration, consider the rhotrix R_5 of Definition i. The principal matrix of R_5 is

$$\begin{pmatrix} a & b & c \\ f & g & h \\ k & l & m \end{pmatrix}.$$

Definition x: The complementary matrix of a rhotrix is a matrix with the largest dimension, whose entries do not coincide with the entries in the principal matrix. It is called *complementary* because coupled with all the entries in the principal matrix, the whole of its entries serves as the complement in making up the entire entries in the rhotrix.

Again, considering the rhotrix R_5 of Definition i, the complementary matrix of R_5 is $\begin{pmatrix} d & e \\ i & j \end{pmatrix}$.

Definition xi: The inscribed matrix of a rhotrix An *inscribed matrix* of a rhotrix is a 'non-tilted' matrix whose entries at the 'four corners' of the matrix all coincide with some of the elements of the boundary of the rhotrix. In other words, any 'non-tilted' matrix in a rhotrix, which is wide enough for its vertices to coincide with some boundary entries of the rhotrix is known as an inscribed matrix of the rhotrix.

The inscribed matrix of R_5 of Definition i is $\begin{pmatrix} f & d & b \\ i & g & e \\ l & j & h \end{pmatrix}$.

Definition xii: Major matrices of a rhotrix In a rhotrix, the principal, complementary and inscribed matrices are called the *major matrices* of the rhotrix. Every other matrix in the rhotrix is called a *minor matrix*. Also every other rhotrix embedded in the rhotrix other than the rhotrix itself is referred to as a minor rhotrix. A rhotrix is called the *host rhotrix* in relation to its major and minor matrices and also in relation to its minor rhotrices. Further examples of the major matrices of a rhotrix are found in the next section.

5 Standardizing major matrices of a rhotrix

In presenting a major matrix – principal, complementary or inscribed matrix – of a rhotrix in an arbitrary form, it is relevant that the matrix be in *standard form* of matrix representation as in the embedded matrices (a_{ij}) and (c_{kl}) of the rhotrix $\langle a_{ij}, c_{kl} \rangle$ in Section 3. In other words, that the first index of an entry and the position of the row of the matrix in which the entry lies correspond and the second index of an entry and the position of the column of the matrix in which the entry lies also correspond. In view of this, we devise a variable transformation rule in the form of an assignment statement of the form:

$$\text{"Let } \beta_{\frac{1}{2}\left[l+\frac{1}{2}[(-1)^{l+1}+1]\right],j} = \rho_{lj}\text{"} \tag{3}$$

used to transform a principal or complementary matrix of an arbitrary rhotrix represented in row-wise form, for all indices $l = 1, 3, 5, \ldots, d$ and $j = 1, 2, \ldots, n$ for the principal matrix of the rhotrix or $l = 2, 4, \ldots, d$-1 and $j = 1, 2, \ldots, n$-1 for the complementary matrix of the rhotrix prior to the transformation; where d is the dimension of the rhotrix, n is the order of the principal matrix and n-1 is the order of the complementary matrix of the rhotrix. In other words, the expression in (3) is a rule which transforms the variables ρ_{lj} of the entries of the principal or complementary matrix of a rhotrix from the representation it inherited from row-wise arbitrary rhotrix representation into standard form of arbitrary matrix representation, called the *standardizing rule for rhotrix's principal and complementary matrices*. After the transformation, the matrix whose entries have been represented by ρ_{lj} in the rhotrix is subsequently represented by β_{ij} in standard form of arbitrary matrix representation, where $i, j = 1, 2, \ldots, n$ for the principal matrix and $i, j = 1, 2, \ldots,$ n-1 for the complementary matrix of the rhotrix.

A similar rule for transforming an inscribed matrix into standard form from the form it inherited from its host rhotrix in row-wise arbitrary rhotrix representation is given by:

$$\text{“Let } \beta_{l-\lambda+1,\ \mu-l+1} = \rho_{lj}\text{”} \tag{4}$$

where λ is the first index of the first entry in the matrix's column containing the entry to be transformed, and μ is the first index of the first entry in the matrix's row containing the entry to be transformed. Again, l and j are the indices in the inscribed matrix prior to the transformation. The rule (4) is called the *standardizing rule for rhotrix's inscribed matrices*.

For instances of transforming major matrices in a rhotrix represented in row-wise form of arbitrary rhotrix representation into standard form, consider the following two rhotrices of dimensions 5 and 7:

$$R_5 = \left\langle \begin{array}{ccccc} & & a_{11} & & \\ & a_{31} & a_{21} & a_{12} & \\ a_{51} & a_{41} & a_{32} & a_{22} & a_{13} \\ & a_{52} & a_{42} & a_{33} & \\ & & a_{53} & & \end{array} \right\rangle, \quad R_7 = \left\langle \begin{array}{ccccccc} & & & b_{11} & & & \\ & & b_{31} & b_{21} & b_{12} & & \\ & b_{51} & b_{41} & b_{32} & b_{22} & b_{13} & \\ b_{71} & b_{61} & b_{52} & b_{42} & b_{33} & b_{23} & b_{14} \\ & b_{72} & b_{62} & b_{53} & b_{43} & b_{34} & \\ & & b_{73} & b_{63} & b_{54} & & \\ & & & b_{74} & & & \end{array} \right\rangle.$$

Using the rule "Let $p_{\frac{1}{2}\left[l+\frac{1}{2}[(-1)^{l+1}+1]\right],j} = a_{lj}$", the principal matrix of R_5 is $\begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$,

while the principal matrix of R_7 is $\begin{pmatrix} p'_{11} & p'_{12} & p'_{13} & p'_{14} \\ p'_{21} & p'_{22} & p'_{23} & p'_{24} \\ p'_{31} & p'_{32} & p'_{33} & p'_{34} \\ p'_{41} & p'_{42} & p'_{43} & p'_{44} \end{pmatrix}$ using the rule

$$\text{“Let } {p'}_{\frac{1}{2}\left[l+\frac{1}{2}[(-1)^{l+1}+1]\right],j} = b_{lj}\text{”.}$$

Using the rule

$$\text{“Let } c_{\frac{1}{2}\left[l+\frac{1}{2}[(-1)^{l+1}+1]\right],j} = a_{lj}\text{”,}$$

the complementary matrix of R_5 is $\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$, while the complementary matrix of R_7 is $\begin{pmatrix} c'_{11} & c'_{12} & c'_{13} \\ c'_{21} & c'_{22} & c'_{23} \\ c'_{31} & c'_{32} & c'_{33} \end{pmatrix}$ by using the rule

$$\text{“Let } c'_{\frac{1}{2}\left[l+\frac{1}{2}[(-1)^{l+1}+1]\right],j} = b_{lj}\text{”}.$$

Using the rule “Let $d_{l-\lambda+1,\mu-l+1} = a_{lj}$” the inscribed matrix of R_5 is

$$\begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix},$$

while on the other hand, the rules “Let $e_{l-\lambda+1,\mu-l+1} = b_{lj}$" and “Let $f_{l-\lambda+1,\mu-l+1} = b_{lj}$" give rise to the inscribed matrices

$$\begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \\ e_{41} & e_{42} & e_{43} \\ e_{51} & e_{52} & e_{53} \end{pmatrix} \text{ and } \begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} \\ f_{21} & f_{22} & f_{23} & f_{24} & f_{25} \\ f_{31} & f_{32} & f_{33} & f_{34} & f_{35} \end{pmatrix}$$

of R_7, respectively.

The following properties are evident:

1. The order of the complementary matrix of a rhotrix is one less than the order of the principal matrix of the rhotrix.

2. The dimension of a rhotrix equals the sum of the order of its principal matrix and the order of its complementary matrix.

3. The inscribed matrix of a rhotrix can be a square matrix or a rectangular matrix.

4. The dimension of a rhotrix is one less than twice the order of its principal matrix.

5. The number of inscribed matrices in a rhotrix represented in row-wise arbitrary rhotrix representation equals the first index of the heart of the rhotrix.

6. Any inscribed matrix of a rhotrix contains the heart of the rhotrix.

7. Any inscribed matrix of a rhotrix is always of odd order.

6 Row-column multiplication formula

In the following theorem, we use two rhotrices of the same dimension and represented in row-wise style of arbitrary rhotrices to develop a formula for their row-column multiplication.

Theorem 1: The row-column product $\langle c_{ij}\rangle$ of two rhotrices $\langle a_{ij}\rangle$ and $\langle b_{ij}\rangle$ of dimension d in row-wise form is established by the equation

$$c_{ij} = \sum_{k=1}^{\frac{1}{2}\left[d-(-1)^i\right]} a_{i,k} b_{2k-\frac{1}{2}\left[(-1)^{i+1}+1\right],j}$$

where $i = 1, 2, \ldots, d$ and $j = 1, 2, \ldots, \frac{1}{2}\left[d-(-1)^i\right]$.

Proof. Let d be an odd positive integer and let A and B denote the d dimensional rhotrices in row-wise form of arbitrary rhotrix representation as follows:

$$A = \left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & a_{31} & a_{21} & a_{12} & & & \\ & & . & a_{41} & a_{32} & a_{22} & . & & \\ & a_{d-2,1} & . & . & . & . & . & a_{1,n-1} & \\ a_{d1} & a_{d-1,1} & a_{d-2,2} & . & . & . & a_{3,n-1} & a_{2,n-1} & a_{1n} \\ & a_{d2} & a_{d-1,2} & . & . & . & a_{4,n-1} & a_{3n} & \\ & & . & . & a_{d-2,n-1} & . & . & & \\ & & & a_{d,n-1} & a_{d-1,n-1} & a_{d-2,n} & & & \\ & & & & a_{dn} & & & & \end{array} \right\rangle,$$

$$B = \left\langle \begin{array}{ccccccccc} & & & & b_{11} & & & & \\ & & & b_{31} & b_{21} & b_{12} & & & \\ & & . & b_{41} & b_{32} & b_{22} & . & & \\ & b_{d-2,1} & . & . & . & . & . & b_{1,n-1} & \\ b_{d1} & b_{d-1,1} & b_{d-2,2} & . & . & . & b_{3,n-1} & b_{2,n-1} & b_{1n} \\ & b_{d2} & b_{d-1,2} & . & . & . & b_{4,n-1} & b_{3n} & \\ & & . & . & b_{d-2,n-1} & . & . & & \\ & & & b_{d,n-1} & b_{d-1,n-1} & b_{d-2,n} & & & \\ & & & & b_{dn} & & & & \end{array} \right\rangle.$$

By the definition of row-column rhotrix multiplication [6], $A \circ B =$

$$\left\langle \begin{array}{ccccc}
 & & a_{11} b_{11}+a_{12}b_{31}+\ldots+a_{1,n-1}b_{d-2,1}+a_{1n}b_{d1} & & \\
 & a_{31} b_{11}+a_{32}b_{31}+\ldots+a_{3,n-1}b_{d-2,1}\ a_{3n}b_{d1} & a_{21} b_{21}+a_{22}b_{41}+\ldots+a_{2,n-1}b_{d-1,1} & a_{11} b_{12}+a_{12}b_{32}+\ldots+a_{1,n-1}b_{d-2,2}+a_{1n}b_{d2} & \\
\bullet & a_{41} b_{21}+\ldots+a_{4n-1}b_{d-1,1} & a_{31} b_{12}+a_{32}b_{32}+\ldots+a_{3,n-1}b_{d-2,2}+a_{3n}b_{d2} & a_{21}b_{22}+\ldots+a_{2,n-1}b_{d-1,2} & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & a_{d-2,1} b_{1,n-1}+a_{d-2,2}b_{3,n-1}+\ldots+a_{d-2,n-1}b_{d-2,n-1}+a_{d-2,n}b_{d,n-1} & \bullet & \bullet \\
 & a_{d1} b_{1,n-1}+a_{d2}b_{3,n-1}+\ldots+a_{d,n-1}b_{d-2,n-1}+a_{dn}b_{d,n-1} & a_{d-1,1} b_{2,n-1}+a_{d-1,2}b_{4,n-1}+\ldots+a_{d-1,n-1}b_{d-1,n-1} & a_{d-2,1} b_{1n}+a_{d-2,2}b_{3n}+\ldots+a_{d-2,n-1}b_{d-2,n}+a_{d-2,n}b_{dn} & \\
 & & a_{d1} b_{1n}+a_{d2}b_{3n}+\ldots+a_{d,n-1}b_{d-2,n}+a_{dn}b_{dn} & &
\end{array} \right\rangle$$

$$\text{Let} \quad A \circ B = C = \left\langle \begin{array}{ccccccccc}
 & & & & c_{11} & & & & \\
 & & & c_{31} & c_{21} & c_{12} & & & \\
 & & . & c_{41} & c_{32} & c_{22} & . & & \\
 & c_{d-2,1} & . & . & . & . & . & c_{1,n-1} & \\
c_{d1} & c_{d-1,1} & c_{d-2,2} & . & . & . & c_{3,n-1} & c_{2,n-1} & c_{1n} \\
 & c_{d2} & c_{d-1,2} & . & . & . & c_{4,n-1} & c_{3n} & \\
 & & . & . & c_{d-2,n-1} & . & . & & \\
 & & & c_{d,n-1} & c_{d-1,n-1} & c_{d-2,n} & & & \\
 & & & & c_{dn} & & & &
\end{array} \right\rangle.$$

By comparing entries we have the following equations:

Entries for the principal matrix of C:

$c_{11} = a_{11}b_{11}+a_{12}b_{31}+\ldots+a_{1,n-1}b_{d-2,1}+a_{1n}b_{d1}$

$c_{12} = a_{11}b_{12}+a_{12}b_{32}+\ldots+a_{1,n-1}b_{d-2,2}+a_{1n}b_{d2}$

$c_{31} = a_{31}b_{11}+a_{32}b_{31}+\ldots+a_{3,n-1}b_{d-2,1}+a_{3n}b_{d1}$

$c_{32} = b_{12}+a_{32}b_{32}+\ldots+a_{3,n-1}b_{d-2,2}+a_{3n}b_{d2}$

. . . .

$c_{d-2,n-1} = a_{d-2,1}\ b_{1,n-1}+a_{d-2,2}b_{3,n-1}+\ldots+ a_{d-2,n-1}b_{d-2,n-1}+ a_{d-2,n}b_{d,n-1}$

$c_{d-2,n} = a_{d-2,1}b_{1n}+a_{d-2,2}b_{3n}+\ldots+a_{d-2,n-1}b_{d-2,n}+a_{d-2,n}b_{dn}$

$c_{d,n-1} = a_{d1}b_{1,n-1}+a_{d2}b_{3,n-1}+\ldots+a_{d,n-1}b_{d-2,n-1}+a_{dn}b_{d,n-1}$

$c_{dn} = a_{d1}b_{1n}+a_{d2}b_{3n}+\ldots+a_{d,n-1}b_{d-2,n}+a_{dn}b_{dn}$

Entries for the complementary matrix of C:

$c_{21} = a_{21}b_{21}+a_{22}b_{41}+\ldots+ a_{2,n-1}b_{d-1,1}$

$c_{22} = a_{21}b_{22}+\ldots+a_{2,n-1}b_{d-1,2}$
$c_{41} = a_{41}b_{21}+\ldots+a_{4n-1}b_{d-1,1}$
. . . .
$c_{d-1,n-1} = a_{d-1,1}\ b_{2,n-1}+a_{d-1,2}b_{4,n-1}+\ldots+ a_{d-1,n-1}b_{d-1,n-1}$.

Note that commas have been introduced between indices in the summations to avoid ambiguity. In more compact forms, we have:

Entries for the principal matrix of *C*:
$c_{11} = \sum_{k=1}^{n} a_{1,k}b_{2k-1,1}$
$c_{12} = \sum_{k=1}^{n} a_{1,k}b_{2k-1,2}$
$c_{31} = \sum_{k=1}^{n} a_{3,k}b_{2k-1,1}$
$c_{32} = \sum_{k=1}^{n} a_{3,k}b_{2k-1,2}$
. . . .
$c_{d-2,n-1} = \sum_{k=1}^{n} a_{d-2,k}b_{2k-1,n-1}$
$c_{d-2,n} = \sum_{k=1}^{n} a_{d-2,k}b_{2k-1,n}$
$c_{d,n-1}= \sum_{k=1}^{n} a_{d,k}b_{2k-1,n-1}$
$c_{dn} = \sum_{k=1}^{n} a_{d,k}b_{2k-1,n}$

Entries for the complementary matrix of *C*:
$c_{21} = \sum_{k=1}^{n-1} a_{2,k}b_{2k,1}$
$c_{22} = \sum_{k=1}^{n-1} a_{2,k}b_{2k,2}$
$c_{41} = \sum_{k=1}^{n-1} a_{4,k}b_{2k,1}$
. . . . . .
$c_{d-1,n-1} = \sum_{k=1}^{n-1} a_{d-1,k}b_{2k,n-1}$

More densely,

$$c_{ij} = \sum_{k=1}^{n} a_{i,k}b_{2k-1,j} \quad (\text{ entries for the principal matrix of } C) \tag{5}$$

$$c_{ij} = \sum_{k=1}^{n-1} a_{i,k}b_{2k,j} \qquad (\text{ entries for the complementary matrix of } C) \tag{6}$$

The differences between the two summations above are the number of summands *n* and the first index 2*k* – 1 of *b* in (5) and the number of summands *n* – 1 and the first index 2*k* of *b* in (6). Now, *i* is odd and the first index for *b* is 2*k* -1 for the entries in the principal matrix of *C*, while *i* is even and the first index for *b* is 2*k* - 0 (i.e., 2*k*) for the entries in the complementary matrix of *C*. Consider the function:

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is odd} \\ 0, & \text{if } x \text{ is even} \end{cases}, \quad \text{defined by} \quad f(x) = \frac{1}{2}\left[(-1)^{x+1}+1\right], \qquad x = 1,\ 2, \ldots .$$

Replacing 1 and 0 with $f(i)$ in the first index of b in (5) and (6) respectively, we have:

$$c_{ij} = \sum_{k=1}^{n} a_{i,k} b_{2k-\frac{1}{2}\left[(-1)^{i+1}+1\right],j} \quad \text{(entries for the principal matrix of } C\text{)} \tag{7}$$

$$c_{ij} = \sum_{k=1}^{n-1} a_{i,k} b_{2k-\frac{1}{2}\left[(-1)^{i+1}+1\right],j}, \quad \text{for the complementary matrix of } C \tag{8}$$

Again, consider the function:

$$g(x) = \begin{cases} n, & \text{if } x \text{ is odd} \\ n-1, & \text{if } x \text{ is even} \end{cases}, \quad \text{defined by} \quad g(x) = \frac{1}{2}\left[2n-1-(-1)^x\right], \quad x = 1,\ 2, \ldots$$

We replace with $g(i)$ the only difference in (7) and (8), which are the total number of summands n and $n-1$, respectively, and obtain the single equation:

$$c_{ij} = \sum_{k=1}^{\frac{1}{2}\left[2n-1-(-1)^i\right]} a_{i,k} b_{2k-\frac{1}{2}\left[(-1)^{i+1}+1\right],j}$$

where $i = 1,\ 2, \ldots, d$ and $j = 1, 2, \ldots, \frac{1}{2}\left[2n-1-(-1)^i\right]$

But n is the order of the principal matrix of C. Removing the entries of this matrix from C leaves us with the complementary matrix of C with order n-1. By the second property in Section 5, $2n - 1 = n + (n - 1) = d$. This gives the result:

$$c_{ij} = \sum_{k=1}^{\frac{1}{2}\left[d-(-1)^i\right]} a_{i,k} b_{2k-\frac{1}{2}\left[(-1)^{i+1}+1\right],j}$$

where $i = 1,\ 2, \ldots, d$ and $j = 1,\ 2, \ldots, \frac{1}{2}\left[d-(-1)^i\right]$. $\square$

7 Row-column multiplication of rhotrices exemplified

We demonstrate the proposed formula for row-column multiplication of rhtorices with the following example.

$$\text{Let } A = \left\langle \begin{array}{ccccc} & & 2 & & \\ & 2 & 1 & 0 & \\ 1 & 4 & 4 & -3 & -5 \\ & 0 & 1 & 2 & \\ & & 10 & & \end{array} \right\rangle \text{ and } B = \left\langle \begin{array}{ccccc} & & 4 & & \\ & 9 & 7 & -5 & \\ 1 & -4 & 6 & 0 & 2 \\ & 0 & 1 & -1 & \\ & & 2 & & \end{array} \right\rangle.$$

Let the product of A and B be denoted by

$$C=\left\langle \begin{array}{ccccc} & & c_{11} & & \\ & c_{31} & c_{21} & c_{12} & \\ c_{51} & c_{41} & c_{32} & c_{22} & c_{13} \\ & c_{52} & c_{42} & c_{33} & \\ & & c_{53} & & \end{array} \right\rangle.$$

Following the formula

$$c_{ij} = \sum_{k=1}^{\frac{1}{2}\left[d-(-1)^{i}\right]} a_{i,k} b_{2k-\frac{1}{2}\left[(-1)^{i+1}+1\right],j}$$

where $i = 1,\ 2, \ldots, d$ and $j = 1,\ 2, \ldots, \frac{1}{2}\left[d-(-1)^{i}\right]$, c_{ij} represent the entries of *C*, the resultant rhotrix on executing the row-column multiplication of the rhotrices *A* and *B*. The rhotrices *A* and *B* are of the fifth dimension and so *d* = 5. Now,

$c_{11}=\sum_{k=1}^{\frac{1}{2}\left[5-(-1)^{1}\right]} a_{1,k} b_{2k-\frac{1}{2}\left[(-1)^{1+1}+1\right],1} = \sum_{k=1}^{3} a_{1,k} b_{2k-1,1} = a_{11}b_{11} + a_{12}b_{31} + a_{13}b_{51} = (2)(4) + (0)(9) + (-5)(1) = 3.$

$c_{12}=\sum_{k=1}^{\frac{1}{2}\left[5-(-1)^{1}\right]} a_{1,k} b_{2k-\frac{1}{2}\left[(-1)^{1+1}+1\right],2} = \sum_{k=1}^{3} a_{1,k} b_{2k-1,2} = a_{11}b_{12} + a_{12}b_{32} + a_{13}b_{52} = (2)(-5) + (0)(6) + (-5)(0) = -10.$

$c_{13}=\sum_{k=1}^{\frac{1}{2}\left[5-(-1)^{1}\right]} a_{1,k} b_{2k-\frac{1}{2}\left[(-1)^{1+1}+1\right],3} = \sum_{k=1}^{3} a_{1,k} b_{2k-1,3} = a_{11}b_{13} + a_{12}b_{33} + a_{13}b_{53} = (2)(2) + (0)(-1) + (-5)(2) = -6.$

$c_{21}=\sum_{k=1}^{\frac{1}{2}\left[5-(-1)^{2}\right]} a_{2,k} b_{2k-\frac{1}{2}\left[(-1)^{2+1}+1\right],1} = \sum_{k=1}^{2} a_{2,k} b_{2k,1} = a_{21}b_{21} + a_{22}b_{41} = (1)(7) + (-3)(-4) = 19.$

$c_{22}=\sum_{k=1}^{\frac{1}{2}\left[5-(-1)^{2}\right]} a_{2,k} b_{2k-\frac{1}{2}\left[(-1)^{2+1}+1\right],2} = \sum_{k=1}^{2} a_{2,k} b_{2k,2} = a_{21}b_{22} + a_{22}b_{42} = (1)(0) + (-3)(1) = -3.$

$c_{31}=\sum_{k=1}^{\frac{1}{2}\left[5-(-1)^{3}\right]} a_{3,k} b_{2k-\frac{1}{2}\left[(-1)^{3+1}+1\right],1} = \sum_{k=1}^{3} a_{3,k} b_{2k-1,1} = a_{31}b_{11} + a_{32}b_{31} + a_{33}b_{51} = (2)(4) + (4)(9) + (2)(1) = 46.$

$c_{32}=\sum_{k=1}^{\frac{1}{2}\left[5-(-1)^{3}\right]} a_{3,k} b_{2k-\frac{1}{2}\left[(-1)^{3+1}+1\right],2} = \sum_{k=1}^{3} a_{3,k} b_{2k-1,2} = a_{31}b_{12} + a_{32}b_{32} + a_{33}b_{52} = (2)(-5) + (4)(6) + (2)(0) = 14.$

$c_{33}=\sum_{k=1}^{\frac{1}{2}\left[5-(-1)^{3}\right]} a_{3,k} b_{2k-\frac{1}{2}\left[(-1)^{3+1}+1\right],3} = \sum_{k=1}^{3} a_{3,k} b_{2k-1,3} = a_{31}b_{13} + a_{32}b_{33} + a_{33}b_{53} = (2)(2) + (4)(-1) + (2)(2) = 4.$

$c_{41}=\sum_{k=1}^{\frac{1}{2}\left[5-(-1)^{4}\right]} a_{4,k} b_{2k-\frac{1}{2}\left[(-1)^{4+1}+1\right],1} = \sum_{k=1}^{2} a_{4,k} b_{2k,1} = a_{41}b_{21} + a_{42}b_{41} = (4)(7) + (1)(-4) = 24.$

$c_{42}=\sum_{k=1}^{\frac{1}{2}\left[5-(-1)^{4}\right]} a_{4,k} b_{2k-\frac{1}{2}\left[(-1)^{4+1}+1\right],2} = \sum_{k=1}^{2} a_{4,k} b_{2k,2} = a_{41}b_{22} + a_{42}b_{42} = (4)(0) + (1)(1) = 1.$

$c_{51}=\sum_{k=1}^{\frac{1}{2}\left[5-(-1)^{5}\right]} a_{5,k} b_{2k-\frac{1}{2}\left[(-1)^{5+1}+1\right],1} = \sum_{k=1}^{3} a_{5,k} b_{2k-1,1} = a_{51}b_{11} + a_{52}b_{31} + a_{53}b_{51} = (1)(4) + (0)(9) + (10)(1) = 14.$

$c_{52} = \sum_{k=1}^{\frac{1}{2}\left[5-(-1)^5\right]} a_{5,k} b_{2k-\frac{1}{2}\left[(-1)^{5+1}+1\right],2} = \sum_{k=1}^{3} a_{5,k} b_{2k-1,2} = a_{51}b_{12} + a_{52}b_{32} + a_{53}b_{52} = (1)(-5) + (0)(6) + (10)(0) = -5.$

$c_{53} = \sum_{k=1}^{\frac{1}{2}\left[5-(-1)^5\right]} a_{5,k} b_{2k-\frac{1}{2}\left[(-1)^{5+1}+1\right],3} = \sum_{k=1}^{3} a_{5,k} b_{2k-1,3} = a_{51}b_{13} + a_{52}b_{33} + a_{53}b_{53} = (1)(2) + (0)(-1) + (10)(2) = 22.$

Hence,

$$C = \left\langle \begin{array}{ccccc} & & 3 & & \\ & 46 & 19 & -10 & \\ 14 & 24 & 14 & -3 & -6 \\ & -5 & 1 & 4 & \\ & & 22 & & \end{array} \right\rangle,$$

which is the row-column product of A and B.

8 The Heart-dimension theorem

The following lemma presents a relationship between the location of the heart of a rhotrix and the order of the principal matrix of the rhotrix. It helps in establishing the formula in Section 9 and, with the *Heart-dimension theorem*, promises to improve the theory in subsequent developments.

Lemma 2. *Let n be the order of the principal matrix of an arbitrary rhotrix R_d in row-wise form of dimension d. The entry a_{pq} is the heart of R_d if and only if $p = n$ and* $q = \frac{1}{2}\left[n + \frac{1}{2}\left[(-1)^{n+1} + 1\right]\right]$.

Proof. Let n be a positive integer and d an odd positive integer. Suppose a_{pq} is the heart of a d-dimensional rhotrix R_d whose principal matrix is of order n. Since the dimension of a rhotrix is one less than twice the order of its principal matrix (see the fourth property in Section 5), then $d = 2n - 1$. Again, since the heart of a rhotrix is located halfway along its major axes, then $p = \frac{d+1}{2}$. It follows that $p = \frac{d+1}{2} = \frac{(2n-1)+1}{2} = \frac{2n}{2} = n$.

Consider the following expressions for q in terms of p for various values of d.
$q = \frac{1}{2}[p+1]$ for the heart a_{11} of R_1
$q = \frac{1}{2}p$ for the heart a_{21} of R_3
$q = \frac{1}{2}[p+1]$ for the heart a_{32} of R_5
$q = \frac{1}{2}p$ for the heart a_{42} of R_7
$q = \frac{1}{2}[p+1]$ for the heart a_{53} of R_9
$q = \frac{1}{2}p$ for the heart a_{63} of R_{11}
$q = \frac{1}{2}[p+1]$ for the heart a_{74} of R_{13}
. . . .

Continuing in this way we obtain an oscillating sequence

$$\left\{\frac{1}{2}[p+t]\right\} \tag{9}$$

in terms of p and t of values of q. Clearly $t = 1$ and p is odd for $d = 1, 5, 9, 13, \ldots$ while $t = 0$ and p is even for $d = 3, 7, 11, 15, \ldots$. Consider the function

$f(x) = \begin{cases} 0, \text{ if } x \text{ is even} \\ 1, \text{ if } x \text{ is odd} \end{cases}$, defined by $f(x) = \frac{1}{2}[(-1)^{x+1}+1]$.

Replacing t with $f(p)$ in (9) we have $q = [p + \frac{1}{2}[(-1)^{p+1}+1]]$, which implies

$$q = [n + \frac{1}{2}[(-1)^{n+1}+1]].$$

Conversely, let $p = n$ and $q = [n + \frac{1}{2}[(-1)^{n+1}+1]]$ for the rhotrix R_d of dimension d whose principal matrix is of order n. To show that a_{pq} is the heart of R_d, it suffices to show that q is the arithmetic mean of the second indices of the entries in the row containing a_{qp}. The sum of the deviations from the arithmetic mean of a distribution is zero [9]. Now n is odd for $d = 1, 5, 9, 13, \ldots$ and $j = 1, 2, 3, \ldots, n$ which implies $q = \frac{1}{2}(n+1)$. Thus, $1 - q + 2 - q + 3 - q + \cdots + (n-1) - q + n - q$ (n many times)

$= 1 - \frac{1}{2}(n+1) + 2 - [\frac{1}{2}(n+1)] + 3 - [\frac{1}{2}(n+1)] + \cdots + (n-1) - [\frac{1}{2}(n+1)]$

$+ n - [\frac{1}{2}(n+1)] = \frac{n^2+n}{2} - \frac{n^2+n}{2} = 0.$

Similarly, n is even for $d = 3, 7, 11, 15, \ldots$ and $j = 1, 2, 3, \ldots, n-1$ which implies $q = \frac{n}{2}$. Thus,

$1 - q + 2 - q + 3 - q + \cdots + (n-2) - q + (n-1) - q$ ($n-1$ many times)

$= \frac{n^2-n}{2} - \frac{n^2-n}{2} = 0.$

It therefore follows that a_{pq} is the heart of R_d. □

In the above proof, it can be noticed that rhotrices of all dimensions do not necessarily have the same properties. In fact, there are two major categories of rhotrices according to the orders of their principal matrices. The rhotrices R_1, R_5, R_9, R_{13}, $\ldots$ have principal matrices whose orders are odd and whose entries include the hearts of their host rhotrices. On the other hand, the rhotrices R_3, R_7, R_{11}, R_{15}, $\ldots$ have principal matrices with even orders. However, the entries in their principal matrices do not include the hearts of their host rhotrices. Rhotrices in the former category are called *rhotrices of the first kind* while the later are called *rhotrices of the second kind*. Below are further differences between the two categories of rhotrices.

RHOTRIX OF THE FIRST KIND	RHOTRIX OF THE SECOND KIND
Principal matrix has an odd order.	Principal matrix has an even order.
Principal matrix contains the heart entry.	Complementary matrix contains the heart entry.
Complementary matrix is of even order.	Complementary matrix is of odd order.
Can have several rectangular inscribed matrices, but with only one square inscribed matrix.	Can have several rectangular inscribed matrices, but no square inscribed matrix.

In the trivial cases, the principal matrix of R_1 has only a single entry and is the same as its inscribed matrix. R_1 has no complementary matrix. On the other hand, the principal matrix of R_3 is obviously a 2×2 square matrix. Also, R_3 has a complementary matrix which has only a single entry, and has no inscribed matrix.

Theorem 3 establishes a relationship between the location of the heart of a rhotrix and the dimension of the rhotrix.

Theorem 3. (Heart-dimension theorem) *The entry a_{pq} is the heart of a rhotrix R_d in row-wise form of dimension d if and only if p = $\frac{d+1}{2}$ and $q = \frac{1}{4}\left[(-1)^{\frac{d+3}{2}} + d + 2\right]$.*

Proof. The proof follows from Lemma 2 above, with $n = \frac{d+1}{2}$. □

The theorem that follows tries, in general, to capture heart-oriented multiplication of rhotrices. The aim is to construct a single formula to represent the multiplication for possible ease of implementation in a computer algorithm.

9 Heart-oriented multiplication formula

Theorem 4. *Let a_{uv} and b_{uv} be the hearts of two arbitrary rhotrices $\langle a_{ij}\rangle$ and $\langle b_{ij}\rangle$ of dimension d, respectively, represented in row-wise form. The heart-oriented multiplication $\langle c_{ij}\rangle$ of $\langle a_{ij}\rangle$ and $\langle b_{ij}\rangle$ is given by $c_{ij} = \frac{(a_{ij}b_{uv}+a_{uv}b_{ij})f(x)}{2}$, where u is the dimension of the principal matrices of the rhotrices, $v = \frac{1}{2}\left[u + \frac{1}{2}\left[(-1)^{u+1} + 1\right]\right]$, $x = (u-i)^2 + (v-j)^2$ for i, j = 1, 2, …d; $f(x) = \ r_0 + 1$, where x is expressed in the form*

$$\sum_{k=0}^{l-1} 2^{l-(k+1)} r_k,\ and\ r_k \in \{0,1\}\ \ for\ k = 0,\ 1,\ 2, \ldots, l-1,$$

l being the minimum positive integer for which the summation holds.

Proof. Let $\langle a_{ij} \rangle$ and $\langle b_{ij} \rangle$ denote, respectively, the two rhotrices

$$A = \left\langle \begin{array}{ccccccccc}
 & & & & a_{11} & & & & \\
 & & & a_{31} & a_{21} & a_{12} & & & \\
 & & . & a_{41} & a_{32} & a_{22} & . & & \\
 & a_{d-2,1} & . & . & . & . & . & a_{1,n-1} & \\
a_{d1} & a_{d-1,1} & a_{d-2,2} & . & . & . & a_{3,n-1} & a_{2,n-1} & a_{1n} \\
 & a_{d2} & a_{d-1,2} & . & . & . & a_{4,n-1} & a_{3n} & \\
 & & . & . & a_{d-2,n-1} & . & . & & \\
 & & & a_{d,n-1} & a_{d-1,n-1} & a_{d-2,n} & & & \\
 & & & & a_{dn} & & & &
\end{array} \right\rangle,$$

$$B = \left\langle \begin{array}{ccccccccc}
 & & & & b_{11} & & & & \\
 & & & b_{31} & b_{21} & b_{12} & & & \\
 & & . & b_{41} & b_{32} & b_{22} & . & & \\
 & b_{d-2,1} & . & . & . & . & . & b_{1,n-1} & \\
b_{d1} & b_{d-1,1} & b_{d-2,2} & . & . & . & b_{3,n-1} & b_{2,n-1} & b_{1n} \\
 & b_{d2} & b_{d-1,2} & . & . & . & b_{4,n-1} & b_{3n} & \\
 & & . & . & b_{d-2,n-1} & . & . & & \\
 & & & b_{d,n-1} & b_{d-1,n-1} & b_{d-2,n} & & & \\
 & & & & b_{dn} & & & &
\end{array} \right\rangle.$$

Let n, u and v be positive integers denoting the order of the principal matrices, the first and the second indices of the hearts of the rhotrices, respectively. By the definition of heart-oriented multiplication of rhotrices [1], we have $A \circ B =$

$$\left\langle \begin{array}{ccccc}
 & & a_{11}b_{uv} + a_{uv}b_{11} & & \\
 & a_{31}b_{uv} + a_{uv}b_{31} & a_{21}b_{uv} + a_{uv}b_{21} & a_{12}b_{uv} + a_{uv}b_{12} & \\
\iddots & a_{41}b_{uv} + a_{uv}b_{41} & a_{32}b_{uv} + a_{uv}b_{32} & a_{22}b_{uv} + a_{uv}b_{22} & \ddots \\
\iddots & \vdots & \vdots & \vdots & \ddots \\
 & & a_{uv}b_{uv} & & \\
 & & \vdots & & \\
\ddots & \vdots & a_{d-2,n-1}b_{uv} + a_{uv}b_{d-2,n-1} & \vdots & \iddots \\
 & a_{d,n-1}b_{uv} + a_{uv}b_{d,n-1} & a_{d-1,n-1}b_{uv} + a_{uv}b_{d-1,n-1} & a_{d-2,n}b_{uv} + a_{uv}b_{d-2,n} & \\
 & & a_{dn}b_{uv} + a_{uv}b_{dn} & &
\end{array} \right\rangle.$$

$$\text{Let}\quad A\circ B = C = \left\langle \begin{array}{ccccccccc}
 & & & & c_{11} & & & & \\
 & & & c_{31} & c_{21} & c_{12} & & & \\
 & & . & c_{41} & c_{32} & c_{22} & . & & \\
 & c_{d-2,1} & . & . & . & . & . & c_{1,n-1} & \\
c_{d1} & c_{d-1,1} & c_{d-2,2} & . & . & . & c_{3,n-1} & c_{2,n-1} & c_{1n} \\
 & c_{d2} & c_{d-1,2} & . & . & . & c_{4,n-1} & c_{3n} & \\
 & & . & . & c_{d-2,n-1} & . & . & & \\
 & & & c_{d,n-1} & c_{d-1,n-1} & c_{d-2,n} & & & \\
 & & & & c_{dn} & & & &
\end{array} \right\rangle.$$

By comparing entries:
For the non-heart entries of the rhotrix C, we have:
$c_{11} = a_{11}b_{uv} + a_{uv}b_{11}$
$c_{12} = a_{12}b_{uv} + a_{uv}b_{12}$
$c_{13} = a_{13}b_{uv} + a_{uv}b_{13}$
. . . .
$c_{dn} = a_{dn}b_{uv} + a_{uv}b_{dn}$.

For the heart entry of the rhotrix C, we have:
$c_{uv} = a_{uv}b_{uv}$ which we can rewrite as

$$c_{ij} = \frac{1}{2}\left(a_{uv}b_{uv} + a_{uv}b_{uv}\right),\ \text{for } i = u \text{ and } j = v.$$

In general, for the non-heart entries of the rhotrix C we have:

$$c_{ij} = \frac{2}{2}\left(a_{ij}b_{uv} + a_{uv}b_{ij}\right)\ \text{for } i \neq u \text{ or } j \neq v.$$

It is clear that $x = 0$ for the heart entry and $x > 0$ for the non-heart entries of C, where $x{=}(u{-}i)^2{+}\,(v{-}j)^2$.

For the non-heart entries of C, let $0 = x_0 < x_1 < x_2 < \ldots < x_{l-1} = x$ be some l non-negative integers and $r_0, r_1, r_2, \ldots, r_{l-2} \in \{0, 1\}$ such that

$$\begin{array}{c}
\frac{x_1}{2} = x_0 + \frac{r_0}{2} \\
\frac{x_2}{2} = x_1 + \frac{r_1}{2} \\
\ldots\ldots \\
\frac{x_{l-1}}{2} = x_{l-2} + \frac{r_{l-2}}{2},
\end{array}$$

l being the smallest non-negative integer for which the above equations hold. Then,

$$\begin{array}{c}
x_1 = 2x_0 + r_0 \\
x_2 = 2x_1 + r_1 \\
x_3 = 2x_2 + r_2 \\
\ldots\ldots\ldots \\
x_{l-1} = 2x_{l-2} + r_{l-2},
\end{array}$$

and it follows that

$$\begin{gathered}
x_1 = 2x_0 + r_0, \\
x_2 = 2(2x_0 + r_0) + r_1, \\
x_3 = 2(2(2x_0 + r_0) + r_1) + r_2, \\
\dots\dots\dots\dots\dots \\
x_{l-1} = 2\left(\dots 2(2\left(2\left(2x_0 + r_0\right) + r_1\right) + r_2\right) + \dots + r_{l-3}) + r_{l-2}.
\end{gathered}$$

Since $x_0 = 0$ it follows that

$$x_{l-1} = 2\left(\dots 2(2\left(2r_0 + r_1\right) + r_2\right) + \dots + r_{l-3}) + r_{l-2},$$

and since $x = x_{l-1}$ then $x = 2\left(\dots 2(2\left(2r_0 + r_1\right) + r_2\right) + \dots + r_{l-3}) + r_{l-2}$, which implies

$$x = \sum_{k=0}^{l-1} 2^{l-(k+1)} r_k.$$

Note that $r_k = 0$ or $r_k = 1$ for $k = 0$. Consider the function

$$f(x) = \begin{cases} 1, & \text{if } x = 0 \\ 2, & \text{if } x > 0 \end{cases},$$

defined by $f(x) = r_0 + 1,$ for all non$-$negative integers x. Substituting $f(x)$ for 1 for the heart entry of the rhotrix C and substituting $f(x)$ for 2 for the non-heart entries of the rhotrix C we get the equation

$$c_{ij} = \frac{(a_{ij}b_{uv} + a_{uv}b_{ij})(r_0 + 1)}{2}$$

or

$$c_{ij} = \frac{(a_{ij}b_{uv} + a_{uv}b_{ij})f(x)}{2},$$

where $x = (u - i)^2 + (v - j)^2,\ i,\ j = 0,\ 1,\ 2, \dots d.$

Also, since n is the dimension of the principal matrix of the rhotrices, by Lemma 2 $u = n$, and it follows that

$$v = \frac{1}{2}\left[u + \frac{1}{2}\left[(-1)^{u+1} + 1\right]\right]$$

and *u* is the dimension of the principal matrices of the rhotrices. $\square$

10 Heart-oriented multiplication of rhotrices exemplified

We demonstrate the above result of Heart-oriented multiplication using the following example.

$$\text{Let } A = \left\langle \begin{array}{ccccc} & & 3 & & \\ & 2 & 1 & 0 & \\ 5 & 4 & 3 & -1 & 1 \\ & 0 & 1 & 2 & \\ & & -1 & & \end{array} \right\rangle \text{ and } B = \left\langle \begin{array}{ccccc} & & 4 & & \\ & 0 & -3 & -5 & \\ 11 & 1 & -2 & 2 & 6 \\ & 0 & 9 & 3 & \\ & & 2 & & \end{array} \right\rangle.$$

Let the product of A and B be denoted by

$$C = \left\langle \begin{array}{ccccc} & & c_{11} & & \\ & c_{31} & c_{21} & c_{12} & \\ c_{51} & c_{41} & c_{32} & c_{22} & c_{13} \\ & c_{52} & c_{42} & c_{33} & \\ & & c_{53} & & \end{array} \right\rangle.$$

Here, $u = 3$, $v = 2$, $d = 5$, $c_{ij} = \frac{(a_{ij}b_{32}+a_{32}b_{ij})f(x)}{2}$.

$c_{11} : x = (3-1)^2 + (2-1)^2 = 5 = \sum_{k=0}^{2} 2^{3-(k+1)} r_k = 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 \Rightarrow r_0 = 1 \Rightarrow f(5) = r_0 + 1 = 1 + 1 = 2 \Rightarrow c_{11} = \frac{2[3(-2)\ +\ (3)4]}{2} = 6.$

$c_{12} : x = (3-1)^2 + (2-2)^2 = 4 = \sum_{k=0}^{2} 2^{3-(k+1)} r_k = 1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0 \Rightarrow r_0 = 1 \Rightarrow f(4) = r_0 + 1 = 1 + 1 = 2 \Rightarrow c_{12} = \frac{2[0(-2)\ +\ (3)(-5)]}{2} = -15.$

c_{13}: $x = (3-1)^2 + (2-3)^2 = 5 = \sum_{k=0}^{2} 2^{3-(k+1)} r_k = 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 \Rightarrow r_0 = 1 \Rightarrow f(5) = r_0 + 1 = 1 + 1 = 2 \Rightarrow c_{13} = \frac{2[1(-2)\ +\ (3)6]}{2} = 16.$

c_{21}: $x = (3-2)^2 + (2-1)^2 = 2 = \sum_{k=0}^{1} 2^{2-(k+1)} r_k = 1 \times 2^1 + 0 \times 2^0 \Rightarrow r_0 = 1 \Rightarrow f(2) = r_0 + 1 = 1 + 1 = 2 \Rightarrow c_{21} = \frac{2[1(-2)\ +\ (3)(-3)]}{2} = -11.$

c_{22}: $x = (3-2)^2 + (2-2)^2 = 1 = \sum_{k=0}^{0} 2^{1-(k+1)} r_k = 1 \times 2^0 \Rightarrow r_0 = 1 \Rightarrow f(1) = r_0 + 1 = 1 + 1 = 2 \Rightarrow c_{22} = \frac{2[(-1)(-2)\ +\ (3)2]}{2} = 8.$

c_{31}: $x = (3-3)^2 + (2-1)^2 = 1 = \sum_{k=0}^{0} 2^{1-(k+1)} r_k = 1 \times 2^0 \Rightarrow r_0 = 1 \Rightarrow f(1) = r_0 + 1 = 1 + 1 = 2 \Rightarrow c_{31} = \frac{2[2(-2)\ +\ (3)0]}{2} = -4.$

c_{32}: $x = (3-3)^2 + (2-2)^2 = 0 = \sum_{k=0}^{0} 2^{1-(k+1)} r_k = 0 \times 2^0 \Rightarrow r_0 = 0 \Rightarrow f(0) = r_0 + 1 = 0 + 1 = 1 \Rightarrow c_{32} = \frac{1[3(-2)\ +\ (3)(-2)]}{2} = -6.$

c_{33}: $x = (3-3)^2 + (2-3)^2 = 1 = \sum_{k=0}^{0} 2^{1-(k+1)} r_k = 1 \times 2^0 \Rightarrow r_0 = 1 \Rightarrow f(1) = r_0 + 1 = 1 + 1 = 2 \Rightarrow c_{33} = \frac{2[2(-2)\ +(3)3]}{2} = 5.$

c_{41}: x = (3-4)^2 + (2-1)^2 = 2 = $\sum_{k=0}^{1} 2^{2-(k+1)} r_k$ = 1 × 2^1 + 0 × 2^0 ⇒ r_0 = 1⇒ $f(2)$ = r_0+1 = 1 + 1 = 2 ⇒ c_{41} = $\frac{2[4(-2)\ +(3)1]}{2}$ = -5.

c_{42}: x = (3-4)^2 + (2-2)^2 = 1 = $\sum_{k=0}^{0} 2^{1-(k+1)} r_k$ = 1 × 2^0 ⇒ r_0 = 1⇒ $f(1)$ = r_0 + 1 = 1 + 1 = 2 ⇒ c_{42} = $\frac{2[1(-2)\ +(3)9]}{2}$ = 25.

c_{51}: x = (3-5)^2 + (2-1)^2 = 5 = $\sum_{k=0}^{2} 2^{3-(k+1)} r_k$ = 1 × 2^2 + 0 × 2^1 + 1 × 2^0 ⇒ r_0 = 1⇒ $f(5)$ = r_0 + 1 = 1 + 1 = 2 ⇒ c_{51}= $\frac{2[5(-2)\ +\ (3)11]}{2}$ = 23.

c_{52}: x = (3-5)^2 + (2-2)^2 = 4 = $\sum_{k=0}^{2} 2^{3-(k+1)} r_k$ = 1 × 2^2 + 0 × 2^1 + 0 × 2^0 ⇒ r_0 = 1⇒ $f(4)$ = r_0 + 1 = 1 + 1 = 2 ⇒ c_{52}= $\frac{2[0(-2)\ +\ (3)0]}{2}$ = 0.

c_{53}: x = (3-5)^2 + (2-3)^2 = 5 = $\sum_{k=0}^{2} 2^{3-(k+1)} r_k$ = 1 × 2^2 + 0 × 2^1 + 1 × 2^0 ⇒ r_0 = 1⇒ $f(5)$ = r_0 + 1 = 1 + 1 = 2 ⇒ c_{53}= $\frac{2[(-1)(-2)\ +\ (3)2]}{2}$ = 8.

Thus, we have

$$C = \left\langle \begin{array}{ccccc} & & 6 & & \\ & -4 & -11 & -15 & \\ 23 & -5 & -6 & 8 & 16 \\ & 0 & 25 & 5 & \\ & & 8 & & \end{array} \right\rangle .$$

C can be seen to be the heart-oriented multiplication of the rhotrices A and B.

11 Subrhotrices and submatrices of a rhotrix

Definition xiii. A rhotrix Q_m of dimension m is called a *subrhotrix* of a rhotrix R_n of dimension n if and only if the following conditions are satisfied:

1. $m \leq n$.
2. Every entry in Q_m is also an entry in R_n.
3. All entries in the same row/column in Q_m cannot be in different rows/columns in R_n, and vice versa.
4. If the respective rows of two entries a and b are i and k in R_n and p and r in Q_m then $i < k \implies p < r$.
5. If the respective columns of two entries a and b are j and l in R_n and q and s in Q_m then $j < l \implies q < s$.

By this definition, it is clear that every rhotrix is a subrhotrix of itself. We now give examples to throw more light on the concept of subrhotrix of a rhotrix, and to verify that the above definition has been reasonably constructed. Consider the following rhotrices of dimensions 3 and 5.

$$P_3 = \left\langle \begin{array}{ccc} & 1 & \\ 7 & 0 & 13 \\ & 2 & \end{array} \right\rangle \text{ and } P_5 = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 7 & 0 & 13 & \\ 1 & 0 & 2 & 7 & 1 \\ & 1 & 13 & 0 & \\ & & 2 & & \end{array} \right\rangle.$$

The dimension of P_3 is less than the dimension of P_5 thereby violating the first condition of Definition xiii. Thus, P_5 cannot be a subrhotrix of P_3 even though it contains all the entries of P_3.

Consider the following rhotrices:

$$R_5 = \left\langle \begin{array}{ccccc} & & 6 & & \\ & 7 & -1 & 13 & \\ 12 & -3 & 2 & 3 & 20 \\ & 10 & 15 & 8 & \\ & & 9 & & \end{array} \right\rangle \text{ and } R_3 = \left\langle \begin{array}{ccc} & 6 & \\ 7 & -1 & 19 \\ & 2 & \end{array} \right\rangle.$$

The entry 19 of R_3 is not an entry in R_5, thereby violating the second condition of Definition xiii. Thus, R_3 is not a subrhotrix of R_5.

Consider the rhotrices

$$R_7 = \left\langle \begin{array}{ccccccc} & & & 11 & & & \\ & & 3 & 6 & 23 & & \\ & -2 & 7 & \mathbf{1} & 13 & 30 & \\ 22 & 12 & \mathbf{2} & \mathbf{1} & \mathbf{-1} & 20 & 4 \\ & 15 & 10 & 4 & 8 & -2 & \\ & & 5 & \mathbf{0} & 31 & & \\ & & & -6 & & & \end{array} \right\rangle \text{ and } R_3 = \left\langle \begin{array}{ccc} & 1 & \\ 2 & 1 & -1 \\ & 0 & \end{array} \right\rangle$$

The relationship between the rhotrices R_7 and R_3 violates the third condition of Definition xiii since 2 and 0 are in the same row in R_3 but are not in the same row in R_7. That is, 2 and 0 are both in the third row of R_3, but 2 is in the fifth row while 0 is in the sixth row of R_7. Thus, R_3 is not a subrhotrix of R_7. If 0 were at the position where 4 is in R_7, then both 2 and 0 would be in the same row (the fifth row) in R_7 and R_3 would be a subrhotrix of R_7. If we consider rewriting R_7 by swapping 0 and 31 so that 0 and 2 are in the same row as in

$$R_7 = \left\langle \begin{array}{ccccccc} & & & 11 & & & \\ & & 3 & 6 & 23 & & \\ & -2 & 7 & \mathbf{1} & 13 & 30 & \\ 22 & 12 & \mathbf{2} & \mathbf{1} & \mathbf{-1} & 20 & 4 \\ & 15 & 10 & 4 & 8 & -2 & \\ & & 5 & 31 & \mathbf{0} & & \\ & & & -6 & & & \end{array} \right\rangle,$$

then R_3 is again not a subrhotrix of R_7 by the third condition of Definition xiii, since 0 and -1 are in the same column in R_3 but are not in the same column in R_7.

Consider the following two rhotrices

$$Q_5 = \left\langle \begin{array}{ccccc} & & \mathbf{7} & & \\ & \mathbf{6} & \mathbf{5} & \mathbf{-3} & \\ 1 & -2 & \mathbf{4} & 7 & 1 \\ & 1 & 13 & 0 & \\ & & 2 & & \end{array} \right\rangle \text{ and } Q_3 = \left\langle \begin{array}{ccc} & 6 & \\ 7 & 5 & 4 \\ & -3 & \end{array} \right\rangle.$$

The row containing 7 and -3 comes before the row containing 6 and 4 in Q_5, whereas the row containing 6 and 4 comes before the row containing 7 and -3 in Q_3, thereby violating the fourth condition of Definition xiii. Hence, Q_3 is not a subrhotrix of Q_5. However, the rhotrix

$$\left\langle \begin{array}{ccc} & 7 & \\ 1 & 5 & -3 \\ & 1 & \end{array} \right\rangle \text{ is a subrhotrix of } Q_5.$$

The fifth condition is the column equivalence of the fourth condition.

The subrhotrices of the rhotrix

$$R_7 = \left\langle \begin{array}{ccccccc} & & & 1 & & & \\ & & 2 & 6 & 3 & & \\ & -11 & 7 & -1 & 13 & -3 & \\ 9 & 12 & 2 & -4 & -1 & 8 & 4 \\ & 2 & 10 & 6 & 3 & -2 & \\ & & -2 & 15 & 0 & & \\ & & & -7 & & & \end{array} \right\rangle.$$

include

$$\left\langle \begin{array}{ccc} & 3 & \\ -1 & 13 & -3 \\ & -1 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 9 & -4 & 4 \\ & -7 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & -1 & \\ 2 & 15 & -2 \\ & -7 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 9 & 12 & 4 \\ & -7 & \end{array} \right\rangle \text{ and}$$

$$\left\langle \begin{array}{ccccc} & & 1 & & \\ & -11 & 6 & 3 & \\ 9 & 12 & 2 & 8 & 4 \\ & 2 & 15 & 0 & \\ & & -7 & & \end{array} \right\rangle.$$

The rhotrix $R_5 = \left\langle \begin{array}{ccccc} & & 1 & & \\ & -11 & 7 & 3 & \\ 9 & 12 & 2 & 13 & 4 \\ & 2 & 10 & 0 & \\ & & -7 & & \end{array} \right\rangle$ violates the third condition of a subrhotrix of R_7 since 7 and 13 are in different rows in R_7.

Definition xiv Let R be a rhotrix. A matrix T is called a submatrix of R if and only if the following properties are satisfied:

1. There exists a major matrix M in R whose dimension is greater than or equal to the dimension of T.

2. Every entry in T is also an entry in M.

3. All entries in the same row/column in T cannot be in different rows/columns in M, and vice versa.

4. If the respective rows of two entries a and b are i and k in M and p and r in T then $i < k \implies p < r$.

5. If the respective columns of two entries a and b are j and l in M and q and s in T then $j < l \implies q < s$.

By this definition, the major matrices of a rhotrix are submatrices of the rhotrix. We give the following examples to demonstrate the concept of submatrix of a rhotrix, and to verify that the definition has been reasonably constructed. Again, let us consider the following rhotrix of dimension 7.

$$R_7 = \left\langle \begin{array}{ccccccc} & & & 3 & & & \\ & & 2 & 6 & 13 & & \\ & -9 & 7 & -1 & 3 & -3 & \\ 8 & 1 & 2 & 1 & 1 & 4 & 4 \\ & 2 & 10 & 6 & 3 & 2 & \\ & & -2 & 8 & 0 & & \\ & & & 6 & & & \end{array} \right\rangle$$

The matrix

$$\begin{pmatrix} 3 & 13 & -3 & 4 & 13 \\ 2 & -1 & 1 & 2 & 2 \\ -9 & 2 & 6 & 0 & 0 \\ 8 & 2 & -2 & 8 & 6 \\ 2 & 8 & 2 & 2 & 6 \end{pmatrix}$$

is not a submatrix of R_7 since its dimension is greater than the dimension of any major matrix (the principal matrix in particular, since both have common entries) of R_7.

Considering the principal matrix of R_7, the matrix $\begin{pmatrix} 3 & 13 & 4 \\ 2 & -1 & 2 \\ 8 & 2 & 10 \end{pmatrix}$ is not a submatrix of R_7 since 10 is not an entry in the principal matrix $\begin{pmatrix} 3 & 13 & -3 & 4 \\ 2 & -1 & 1 & 2 \\ -9 & 2 & 6 & 0 \\ 8 & 2 & -2 & 6 \end{pmatrix}$ of R_7. This violates the second condition of Definition xiv. Considering the inscribed matrix $\begin{pmatrix} -9 & 7 & -1 & 3 & -3 \\ 1 & 2 & 1 & 1 & 4 \\ 2 & 10 & 6 & 3 & 2 \end{pmatrix}$ of R_7, the matrix $\begin{pmatrix} -9 & 7 \\ 1 & 2 \\ 2 & 3 \end{pmatrix}$ is not a submatrix of R_7 since 3 is in a different column in the inscribed matrix. Similarly, the matrix $\begin{pmatrix} 1 & 2 & 1 & 1 & -3 \\ 2 & 10 & 6 & 3 & 2 \end{pmatrix}$ is not a submatrix of R_7 since -3 is in a different row in the inscribed matrix. Both are violations of the fourth condition of Definition xiv. Considering the complementary matrix $\begin{pmatrix} 6 & 3 & 4 \\ 7 & 1 & 3 \\ 1 & 10 & 8 \end{pmatrix}$ of R_7, the matrix $T = \begin{pmatrix} 1 & 8 \\ 6 & 4 \end{pmatrix}$ is not a submatrix of R_7 since the row containing 6 precedes the row containing 1 in R_7 while the row containing 1 precedes the row containing 6 in T.

12 Conclusion

In this paper, we present some methods of representing arbitrary rhotrices, of particular interest is the row-wise method with which we have established some formulae for rhotrix multiplication. Sample examples have also been included in the present work to demonstrate the application of the presented formulae. The concepts of subrhotrix and submatrix of a rhotrix promises to pave way for a new dimension in describing various algebraic properties of rhotrices. The full exploitation, possible works of algebra and some applications of skew rhotrices have been left out for future research.

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References

[1] Ajibade, A. O., The Concept of Rhotrix in Mathematical Enrichment, *International Journal of Mathematical Education in Science and Technology*, Vol. 34, 2003, No. 2, 175–179.

[2] Atanassov, K. T., A. G. Shannon, Matrix-tersions and Matrix-noitrets: Exercise in Mathematical Enrichment, *Int. J. Math. Educ. Sci. Technol.*, Vol. 29, 1998, 898–903.

[3] Ezugwu, E. A., B. Sani, J. B. Sahalu, The Concept of Heart Oriented Rhotrix Multiplication, *GJSFR*, Vol. 11, 2011, Issue 2, 35–46.

[4] Kaurangini, M. L., B. Sani,, Hilbert Matrix and its Relationship with a Special Rhotrix, *ABACUS (Journal of Mathematical Association of Nigeria),* Vol. 34, 2007, No. 2A, 101–106.

[5] Sani, B., An Alternative Method for Multiplication of Rhotrices, *International Journal of Mathematics Education in Science and Technology,* Vol. 35, 2004, No. 5, 777–781.

[6] Sani, B., The row-column Multiplication of High Dimensional rhotrices, *International Journal of Mathematical Education in Science and Technology*, Vol. 38, 2007, 657–662.

[7] Sani, B., Solution of Two Coupled Matrices, *The Journal of the Mathematical Association of Nigeria*, Vol. 36, 2009, No. 2, 53–57.

[8] Singh, D., C. M. Peter, Multiset-based Tree Model for Membrane Computing, *Computer Science Journal of Moldova*, Vol. 19, 2011, No. 1(55), 3–28.

[9] Valpole, R. E., R. H. Myers, S. L. Myers, *Probability and Statistics for Engineers and Scientists*, Pearson Educational International, London, 2007.