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Note on φ, ψ and σ-functions. Part 5

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Abstract: Inequalities connecting φ, ψ and σ-functions are formulated and proved.
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The paper is a continuation of [1, 2, 3, 4]. Let us define for the natural number $n \geq 2$:

$$n = \prod_{i=1}^{k} p_i^{\alpha_i},$$

where k, α_1, α_2, ... $\alpha_k \geq 1$ are natural numbers and $p_1, p_2, ..., p_k$ are different prime numbers, the following well-known functions (see, e.g. [5])

$$\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}.(p_i - 1), \text{ and } \varphi(1) = 1,$$

$$\psi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}.(p_i + 1), \text{ and } \psi(1) = 1,$$

$$\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i + 1} - 1}{p_i - 1}, \text{ and } \sigma(1) = 1,$$

$$\omega(n) = \sum_{i=1}^{k} \alpha_i, \text{ and } \omega(1) = 1,$$

$$\underline{cas}(n) = k \text{ and } \underline{cas}(1) = 0,$$

$$\underline{set}(n) = \{p_1, p_2, ..., p_k\} \text{ and } \underline{set}(1) = \emptyset.$$

Let

$$\Delta(n) = \omega(n) - \underline{cas}(n).$$

Theorem 1: For every natural number $n \geq 2$,

$$\sigma(n) - \psi(n) \geq (cas(n) - 1)\underline{min}(n)^{\Delta(n)}. \tag{1}$$

Proof: When n is a prime number, then obviously, $\Delta(n) = 0$ and

$$\sigma(n) - \psi(n) = 0 = (\underline{cas}(n) - 1)\underline{min}(n)^{\Delta(n)}.$$

Let us assume that the assertion is valid for some natural number n, such that $\omega(n) = s \geq 1$ for some natural number s. Let p be a prime number. Therefore, for number np, the equality $\omega(np) = \omega(n) + 1 = s + 1$ holds and we will prove the validity of (1) for np. Let

$$X \equiv \sigma(np) - \psi(np) - (\underline{cas}(np) - 1)\underline{min}(np)^{\Delta(np)}.$$

For p there are two cases.

Case 1. $p \notin \underline{set}(n)$. Then $\underline{cas}(np) = \underline{cas}(n) + 1$ and

$$\Delta(np) = \omega(np) - \underline{cas}(np) = \omega(n) + 1 - \underline{cas}(n) - 1 = \Delta(n).$$

If $p \geq \underline{min}(n)$, then $\underline{min}(np) = \underline{min}(n)$.

If n is a prime number, then $\underline{cas}(n) = 1$, $\min(n) = n$, $\underline{cas}(n) = 1$, $\Delta(n) = 0$ and

$$X = \sigma(n)(p+1) - \psi(n)(p+1) - 0.n^0 = (n+1)(p+1) - (n+1)(p+1) = 0.$$

If $n = q^s$ for some prime number q and $s \geq 2$, we see that $\underline{cas}(n) = 1$, $\min(n) = q$, $\Delta(n) = s - 1$ and

$$X = \sigma(n)(p+1) - \psi(n)(p+1) - q^{\Delta(n)} = (p+1)(q^s + q^{s-1} + ... + 1 - q^s - q^{s-1}) - q^{\Delta(n)}$$

$$= (p+1)(q^{s-2} + ... + 1) - q^{s-1} \geq (p+1)q^{s-2} - q^{s-1} > 0.$$

Let $s \geq 2$ and $\underline{cas}(n) \geq 2$. Then

$$X = \sigma(n)(p+1) - \psi(n)(p+1) - \underline{cas}(n)\underline{min}(n)^{\Delta(n)} = (p+1)(\sigma(n) - \psi(n)) - cas(n)\underline{min}(n)^{\Delta(n)}$$

(by assumption)

$$\geq (p+1)(\underline{cas}(n) - 1)\underline{min}(n)^{\Delta(n)} - \underline{cas}(n)\underline{min}(n)^{\Delta(n)}$$

$$= p.\underline{cas}(n)\underline{min}(n)^{\Delta(n)} + \underline{cas}(n)\underline{min}(n)^{\Delta(n)} - (p+1)\underline{min}(n)^{\Delta(n)} - \underline{cas}(n)\underline{min}(n)^{\Delta(n)}$$

$$= p.\underline{cas}(n)\underline{min}(n)^{\Delta(n)} - (p+1)\underline{min}(n)^{\Delta(n)}$$

$$\geq (2p - p - 1)\underline{min}(n)^{\Delta(n)} \geq 0.$$

If $p < \underline{min}(n)$, then $\underline{min}(np) = p$ and

$$X = \sigma(n)(p+1) - \psi(n)(p+1) - \underline{cas}(n)p^{\Delta(n)} = (p+1)(\sigma(n) - \psi(n)) - \underline{cas}(n)p^{\Delta(n)}$$

$$> (p+1)(\sigma(n) - \psi(n)) - \underline{cas}(n)\underline{min}(n)^{\Delta(n)}$$

(by assumption)

$$> (p+1)(\underline{cas}(n) - 1)\underline{min}(n)^{\Delta(n)} - \underline{cas}(n)\underline{min}(n)^{\Delta(n)}$$

$$> ((p+1)(\underline{cas}(n) - 1) - \underline{cas}(n))\underline{min}(n)^{\Delta(n)}$$

$$> (3(\underline{cas}(n) - 1) - \underline{cas}(n))\underline{min}(n)^{\Delta(n)} > 0,$$

because $\underline{cas}(n) \geq 2$.

Case 2. $p \in \underline{set}(n)$. Then $n = p^a m$ for some natural numbers $a, m \geq 1$, $(p, m) = 1$ and

$$\sigma(np) = \sigma(p^{a+1}m) = \frac{p^{a+2} - 1}{p - 1}\sigma(m) = \frac{p^{a+2} - 1}{p^{a+1} - 1}\sigma(p^a m) > p\sigma(n).$$

In addition, $\underline{cas}(np) = \underline{cas}(n)$ and

$$\Delta(np) = \omega(np) - \underline{cas}(np) = \omega(n) + 1 - \underline{cas}(n) = \Delta(n) + 1.$$

If $p \geq \underline{min}(n)$, then $\underline{min}(np) = \underline{min}(n)$ and

$$X = \sigma(np) - \psi(np) - (\underline{cas}(n) - 1)\underline{min}(np)^{\Delta(np)} > p(\sigma(n) - \psi(n)) - (\underline{cas}(n) - 1)\underline{min}(n)^{\Delta(n)+1}$$

(by assumption)

$$\geq p(\underline{cas}(n) - 1)\underline{min}(n)^{\Delta(n)} - (\underline{cas}(n) - 1)\underline{min}(n)^{\Delta(n)+1}$$

$$= (\underline{cas}(n) - 1)\underline{min}(n)^{\Delta(n)}(p - \underline{min}(n)) \geq 0.$$

If $p < \underline{min}(n)$, then $\underline{min}(np) = p$ and

$$X = \sigma(np) - \psi(np) - (\underline{cas}(np) - 1)p^{\Delta(np)} > p(\sigma(n) - \psi(n)) - (\underline{cas}(np) - 1)p^{\Delta(np)}$$

(by assumption)

$$\geq p(\underline{cas}(n) - 1)\underline{min}(n)^{\Delta(n)} - (\underline{cas}(n) - 1)p^{\Delta(n)+1}$$

$$= p(\underline{cas}(n) - 1)(\underline{min}(n)^{\Delta(n)} - p^{\Delta(n)}) > 0.$$

Therefore, (1) is valid.

Let us have the sequence $a_1, a_2, ..., a_n$ with real numbers, such that $a_1 \geq 2$ and for every i $(1 \leq i \leq n-1):$ $a_{i+1} > a_i + 1$. First, we see, that $a_2 \geq a_1 + 1 > 3$. If we assume that for some $n \geq 1$:

$$a_n > n + 1, \tag{2}$$

then

$$a_{n+1} > a_n + 1 > n + 2.$$

Therefore, (2) is valid by induction.

Second, we prove that for the above sequence it is valid:

$$1 - \prod_{i=1}^{n} \frac{a_i - 1}{a_i} \geq \frac{n}{a_n}. \tag{3}$$

For $n = 1$ (3) has the form:

$$1 - \frac{a_1 - 1}{a_1} = \frac{1}{a_1}.$$

Let us assume that (3) is valid for some $n \geq 1$. Then

$$1 - \prod_{i=1}^{n+1} \frac{a_i - 1}{a_i} - \frac{n+1}{a_{n+1}}$$

$$= 1 - \frac{a_{n+1}-1}{a_{n+1}}.\prod_{i=1}^{n}\frac{a_i-1}{a_i} - \frac{n+1}{a_{n+1}}$$

$$> 1 - \frac{a_{n+1}-1}{a_{n+1}}.(1 - \frac{n}{a_n}) - \frac{n+1}{a_{n+1}}$$

$$= 1 - \frac{a_{n+1}-1}{a_{n+1}} + \frac{a_{n+1}-1}{a_{n+1}}.\frac{n}{a_n}) - \frac{n+1}{a_{n+1}}$$

$$= \frac{1}{a_n a_{n+1}}(a_n a_{n+1} - (a_n a_{n+1} - a_n) + n a_{n+1} - n - (n+1)a_n)$$

$$= \frac{1}{a_n a_{n+1}}(a_n + n a_{n+1} - n - (n+1)a_n)$$

$$= \frac{1}{a_n a_{n+1}}(n(a_{n+1} - n - a_n) - n)$$

$$> \frac{1}{a_n a_{n+1}}(n.1 - n) = 0.$$

Therefore, (3) is valid.

Now, we can prove

Theorem 2: For each natural number $n > 1$:

$$n - \varphi(n) > \frac{n}{\underline{max}(n)}.\underline{cas}(n). \tag{4}$$

Proof: For the fixed $n > 1$ we have

$$n - \varphi(n) = n(1 - \prod_{i=1}^{k}\frac{p_i - 1}{p_i}) > n.\frac{k}{p_k},$$

because $p_1 = 2 > 1$ and for every i $(1 \le i \le n) : p_{i+1} \ge p_i + 2$, i.e., the conditions for inequality (3) are satisfy.

From inequality

$$\psi(n) + \varphi(n) \ge 2n$$

for every natural number n and form (4) it follows

$$\psi(n) - n \ge n - \varphi(n) > \frac{n}{\underline{max}(n)}.\underline{cas}(n)$$

that is a new proof of Theorem 2 [4]. Now, we obtain

$$\psi(n) - \varphi(n) = (\psi(n) - n) + (n - \varphi(n)) \ge 2\frac{n}{\underline{max}(n)}.\underline{cas}(n) \tag{5}$$

that is a new proof of Theorem 1 [4].

Finally, from (1) and (5) we obtain

$$\sigma(n) - \varphi(n) = (\sigma(n) - \psi(n)) + (\psi(n) - \varphi(n))$$

$$\ge (cas(n) - 1)\underline{min}(n)^{\Delta(n)} + 2\frac{n}{\underline{max}(n)}.\underline{cas}(n).$$

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