Notes on Number Theory and Discrete Mathematics
Vol. 18 (2012), No. 1, 22–28

Double inequalities on means via quadrature formula

K. M. Nagaraja[1] and P. Siva Kota Reddy[2*]

[1] Department of Mathematics, Sri Krishna Institute of Technology
Bangalore-560 090, India

[2] Department of Mathematics, Acharya Institute of Technology
Bangalore-560 090, India
e-mail: pskreddy@acharya.ac.in
* Corresponding author

Abstract: In this paper, using Simpson's quadrature formula and Jensen inequality for convex function, we obtained some double inequalities among various means.
Keywords: Inequality, Simpson's rule, Convex function, Jensen inequality.
AMS Classification: 25D15.

1 Introduction

Several eminent researchers explored the well known means respectively called Arithmetic mean, Geometric mean and Harmonic mean in the literature in different verticals, these means respectively given by [1, 2];

For $a, b > 0$, then

$$A(a,b) = \frac{a+b}{2}, \qquad G(a,b) = \sqrt{ab} \qquad and \qquad H(a,b) = \frac{2ab}{a+b}.$$

In [3], the authors defined Oscillatory mean and its dual form and they obtained some interesting results.

For $a, b > 0$ and $\alpha \in (0,1)$, then Oscillatory mean and its dual form are as follows;

$$O(a,b;\alpha) = \alpha G(a,b) + (1-\alpha)A(a,b) \tag{1.1}$$

and

$$O^{(d)}(a,b;\alpha) = G(a,b)^{\alpha} A(a,b)^{1-\alpha}. \tag{1.2}$$

For $a, b > 0$, then Seiffert's mean is given by [1, 6];

$$P(a,b) = \frac{b-a}{2\tan^{-1}\left(\frac{b-a}{b+a}\right)} \tag{1.3}$$

For $a, b > 0$ and r is a real number, then the power mean is given by [1];

$$M_r(a,b) = \begin{cases} \left(\frac{a^r+b^r}{2}\right)^{\frac{1}{r}}, & r \neq 0 \\ \sqrt{ab}, & r = 0 \end{cases} \tag{1.4}$$

Let $n \geq 1$ be a fixed natural number and I an interval of real numbers, then for every $a = (a_1, a_2, ...a_n) \in I^n$, the arithmetic mean associated to a is defined as;

$$A_n[a] = \frac{a_1 + a_2 + ... + a_n}{n}.$$

Let $I \in R$ be an interval. If $f : I \to R$ is a convex(concave)function, then the well known Jensen inequality says that;

$$f\left(\frac{a_1 + a_2 + ... + a_n}{n}\right) \leq (\geq) \left(\frac{f(a_1) + f(a_2) + ... + f(a_n)}{n}\right),$$

which can also written in the following notation:

$$f(A_n[a]) \leq (\geq) A_n[f(a)] \tag{1.5}$$

If s and t are two real parameters, a and b are positive numbers $a \neq b$, then the extended means of s, t of a and b is given by [1];

$$G_{s,t}(a,b) = \begin{cases} \left(\frac{a^s+b^s}{a^t+b^t}\right)^{\frac{1}{s-t}}, & if s \neq t \\ \exp\left(\frac{a^s \log a + b^s \log b}{a^s+b^s}\right)^{\frac{1}{s}}, & if s = t, \end{cases} \tag{1.6}$$

and

$$E_{s,t}(a,b) = \begin{cases} \left(\frac{t(a^s-b^s)}{s(a^t-b^t)}\right)^{\frac{1}{s-t}}, & if (s-t)st \neq 0, a \neq b \\ \exp\left(-\frac{1}{s} + \frac{a^s \log a - b^s \log b}{a^s-b^s}\right), & if s = t \neq 0, a \neq b \\ \exp\left(\frac{a^s-b^s}{s(a^s \log a - b^s \log b)}\right)^{\frac{1}{s}}, & if s \neq 0, t = 0, a \neq b \\ \sqrt{ab}, & if s = t = 0 \\ a & if a = b \end{cases} \tag{1.7}$$

are respectively called the Gini means and the Stolarsky means.

Some particular cases of the Gini means and the Stolarsky means in intergal form are given below.

For $t = 0$, the Gini mean $G_{s,0}(a,b)$ coincides with the Holder mean of order $s > 0$ and for $s = 1$, is an Arithmetic mean of a and b.

$$A_{s,0}(a,b) = \left(\frac{a^s + b^s}{2}\right)^{\frac{1}{s}} = \left(\frac{s}{b^s - a^s}\int_a^b x^{2s-1}dx\right)^{\frac{1}{s}},$$

for $s = t = 0$, the Gini mean $G_{0,0}(a,b)$ coincides with the Geometric mean of a and b.

$$G(a,b) = \sqrt{ab} = \left(\frac{1}{b-a}\int_a^b \frac{1}{x^2}dx\right)^{\frac{-1}{2}},$$

for $s=1, t=0$, the Stolarsky mean $E_{1,0}(a,b)$ coincides with the Logarithmic mean of a and b.

$$L(a,b)=\frac{b-a}{\ln b-\ln a}=\left(\frac{1}{b-a}\int_a^b \frac{1}{x}dx\right)^{-1}$$

and for $s=t=1$, the Stolarsky mean $E_{1,1}(a,b)$ coincides with the Identric mean of a and b.

$$I(a,b)=\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}=\exp\left(\frac{1}{b-a}\int_a^b \ln x dx\right).$$

This paper is based on certain inequalities satisfied by the 4-convex functions and Jensen inequality ([2], [4], [5]), that is the functions which are differentiable 4-times and $f^{(4)}(x)\geq 0$ for all values of x. Now recall the Simpson's quadrature formula in the form of the lemma as below.

Lemma: 1.1 *If $f\in C^4([a,b])$ and $f^{(4)}(x)\geq 0$, then the mean value of f*

$$M(f)=\frac{1}{b-a}\int_a^b f(x)dx$$

does not exceed the sum

$$\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$

that is

$$\frac{1}{b-a}\int_a^b f(x)dx=\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{(b-a)^4}{2880}f^{(4)}(c),$$

for some $c\in(a,b)$.

2 Applications to some inequalities among means

In this section, some double inequalities involving important means are established by using Simpson's quadrature rule and Jensen inequality.

Theorem: 2.1 *If $a,b>0$, then holds the following inequality.*

$$G^2(a,b)\leq\left[\frac{2H^2(a,b)+M_2^2(a,b)}{3}\right]\leq M_2^2(a,b).$$

Proof: According to Simpson's quadrature formula,

$$\frac{1}{b-a}\int_a^b f(x)dx=\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{(b-a)^4}{2880}f^{(4)}(c),$$

for some $c\in(a,b)$.

Take $f(x)=\frac{1}{x^2}$, from which $f^4(x)=\frac{120}{x^6}>0$, that is $f^{(4)}(c)=\frac{120}{c^6}>0$, for some $c\in(a,b)$, then

$$\frac{1}{b-a}\int_a^b \frac{1}{x^2}dx\leq\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] \tag{2.1}$$

After simple integration and simplification gives,

$$\frac{1}{G^2(a,b)} \leq \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]. \tag{2.2}$$

since $f(x) = \frac{1}{x^2}$, from which $f^{(2)}(x) = \frac{6}{x^4} > 0$, for all $x \in (a, b)$, hence $f(x)$ is convex function.

The well known Jensen inequality for convex functions says that;

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2},$$

then

$$\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \leq \frac{1}{6}\left[f(a) + 4\left(\frac{f(a)+f(b)}{2}\right) + f(b)\right]. \tag{2.3}$$

By combining inequalities (2.2) and (2.3) leads to,

$$\frac{1}{G^2(a,b)} \leq \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \leq \frac{1}{6}\left[f(a) + 2\left[f(a)+f(b)\right] + f(b)\right] \tag{2.4}$$

Replace $f(a) = \frac{1}{a^2}$, $f(b) = \frac{1}{b^2}$ and $f(\frac{a+b}{2}) = \frac{1}{(\frac{a+b}{2})^2}$, in equation (2.4) gives,

$$\frac{1}{G^2(a,b)} \leq \frac{1}{6}\left[\frac{1}{a^2} + 4\frac{1}{(\frac{a+b}{2})^2} + \frac{1}{b^2}\right] \leq \frac{1}{6}\left[\frac{1}{a^2} + 2\left(\frac{1}{a^2} + \frac{1}{b^2}\right) + \frac{1}{b^2}\right]$$

on rearranging leads to,

$$\frac{1}{G^2(a,b)} \leq \frac{1}{6}\left[\frac{1}{a^2} + \frac{4}{A^2(a,b)} + \frac{1}{b^2}\right] \leq \frac{1}{6}\left[3\left(\frac{1}{a^2} + \frac{1}{b^2}\right)\right]$$

on substituting $a^2b^2 = G^4(a,b)$ and $a^2+b^2 = 2M_2^2(a,b)$, the above inequality takes the following form,

$$\frac{G^4(a,b)}{G^2(a,b)} \leq \frac{1}{3}\left[M_2^2(a,b) + 2\frac{G^4(a,b)}{A^2(a,b)}\right] \leq M_2^2(a,b) \tag{2.5}$$

Further, from the well known identity,

$$G^2(a,b) = A(a,b)H(a,b)$$

on substituting in the equation (2.5) takes the form,

$$G^2(a,b) \leq \left[\frac{2H^2(a,b) + M_2^2(a,b)}{3}\right] \leq M_2^2(a,b). \tag{2.6}$$

This completes the proof of Theorem 2.1.

Note 1: In alternative form the double inequality (2.6) can be expressed as:

$$G^2(a,b) \leq \left[\frac{2H^2(a,b) + A(a^2,b^2)}{3}\right] \leq A(a^2,b^2).$$

Theorem: 2.2 *If $a, b > 0$, then the following inequality holds:*

$$H(a,b) \leq \left[\frac{L(a,b)A(a,b) + 2H(a,b)L(a,b)}{3A(a,b)}\right] \leq L(a,b).$$

Proof: Take $f(x) = \frac{1}{x}$, for which $f^{(4)}(x) = \frac{24}{x^5} > 0$, for all $x \in (a,b)$, since $a,b > 0$. that is $f^{(4)}(c) = \frac{24}{c^5} > 0$, for some $c \in (a,b)$, then

Also for $f(x) = \frac{1}{x}$, from which $f^{(2)}(x) = \frac{2}{x^3} > 0$, for all $x \in (a,b)$, hence $f(x)$ is convex function.

Thus for $f(x) = \frac{1}{x}$, the equations (2.1) and (2.3) together takes the following form;

$$\frac{1}{b-a}\int_a^b \frac{1}{x}dx \leq \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \leq \frac{1}{6}\left[f(a) + 4\left(\frac{f(a)+f(b)}{2}\right) + f(b)\right] \tag{2.7}$$

After calculus and replacing $f(a) = \frac{1}{a}$, $f(b) = \frac{1}{b}$ and $f(\frac{a+b}{2}) = \frac{1}{\left(\frac{a+b}{2}\right)}$, in equation (2.7) becomes,

$$\frac{\ln b - \ln a}{b-a} \leq \frac{1}{6}\left[\frac{1}{a} + \frac{4}{\left(\frac{a+b}{2}\right)} + \frac{1}{b}\right] \leq \frac{1}{2}\left[\frac{1}{a} + \frac{1}{b}\right]$$

is equivalently,

$$\frac{1}{L(a,b)} \leq \frac{1}{6}\left[\frac{2A(a,b)}{G^2(a,b)} + \frac{4}{A(a,b)}\right] \leq \frac{A(a,b)}{G^2(a,b)}$$

use the well known identity,

$$G^2(a,b) = A(a,b)H(a,b)$$

in the above inequality leads to,

$$H(a,b) \leq \left[\frac{L(a,b)A(a,b) + 2H(a,b)L(a,b)}{3A(a,b)}\right] \leq L(a,b).$$

This completes the proof of Theorem 2.2.

Theorem: 2.3 *If $a,b > 0$, then the following inequality holds:*

$$I(a,b) \geq O^{(d)}(a,b;\frac{1}{3}) \geq G(a,b).$$

Proof: Let $f(x) = \ln x$, then $f^{(2)}(x) = \frac{-1}{x^2} < 0$ and $f^{(4)}(x) = \frac{-6}{x^4} < 0$, for all $x \in (a,b)$, hence $f(x)$ is concave function.

Thus for $f(x) = \ln x$, the equations (2.1) and (2.3) together written as,

$$\frac{1}{b-a}\int_a^b \ln x dx \geq \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \geq \frac{1}{6}\left[f(a) + 4\left(\frac{f(a)+f(b)}{2}\right) + f(b)\right] \tag{2.8}$$

After simple calculus and replacing $f(a) = \ln a$, $f(b) = \ln b$ and $f(\frac{a+b}{2}) = \ln(\frac{a+b}{2})$, then the equation (2.8) takes the form,

$$\left[\frac{b\ln b - a\ln a}{b-a} - 1\right] \geq \frac{1}{6}\left[ln\left(ab\left[\frac{a+b}{2}\right]^4\right)\right] \geq \frac{1}{2}\ln(ab)$$

is equivalently after canceling $\ln$ both sides leads to,

$$I(a,b) \geq G^{\frac{1}{3}}(a,b)A^{\frac{1}{3}}(a,b) \geq G(a,b) \tag{2.9}$$

From the definition of dual oscillatory mean, the inequality (2.9) can be rewritten for $\alpha = \frac{1}{3}$ as;

$$I(a,b) \geq O^{(d)}(a,b;\frac{1}{3}) \geq G(a,b).$$

This completes the proof of Theorem 2.3.

Lemma: 2.2 *If a, $b > 1$, then $P(a,b) > L(a,b)$.*

Proof: From the definitions of Logarithmic mean and Seiffert's mean, gives

$$\frac{1}{L(a,b)} - \frac{1}{P(a,b)} = \frac{\ln b - \ln a}{b-a} - \frac{2\tan^{-1}\left(\frac{b-a}{b+a}\right)}{b-a} \tag{2.10}$$

put $b = t+1, a = 1$ in the equation (2.11), then

$$\frac{1}{L(a,b)} - \frac{1}{P(a,b)} = \frac{1}{t}\left[\ln(t+1) - 2\tan^{-1}\left(\frac{t}{t+2}\right)\right] \tag{2.11}$$

let $h(t) = \left[\ln(t+1) - 2\tan^{-1}\left(\frac{t}{t+2}\right)\right]$, then $h'(t) = \frac{2t^2}{(t+1)(2t^2+4t+4)} > 0$. This shows that $h(t)$ is increasing function for $t > 0$, then $\frac{1}{L(a,b)} - \frac{1}{P(a,b)} > 0$, this proves that $P(a,b) > L(a,b)$. Hence the proof of lemma 2.2.

Corollary: 2.1 *If a, $b > 1$, then $\ln\left[\frac{eI(a,b)}{G(a,b)}\right] > \frac{A(a,b)}{P(a,b)}$.*

Proof: The relation between Logarithmic mean and Identric mean is,

$$\ln I(a,b) = \frac{a}{L(a,b)} + \ln b - 1 \quad \texttt{and} \quad \ln I(a,b) = \frac{b}{L(a,b)} + \ln a - 1$$

on adding gives,

$$\ln I(a,b) = \frac{A(a,b)}{L(a,b)} + \ln G(a,b) - 1 \tag{2.12}$$

with simple computations and using lemma 2.2, the above inequality takes the form;

$$\ln\left[\frac{eI(a,b)}{G(a,b)}\right] > \frac{A(a,b)}{P(a,b)}.$$

Hence the proof of corollary 2.1.

Theorem: 2.4 *If $a, b > 0$, then the following inequality holds:*

$$E_{s,1}^{t-1}(a,b) \leq \left[\frac{A(a^{t+1}, b^{t+1}) + 2H^{t+1}(a,b)}{3G^2(a,b)}\right] \leq A(a^{t+1}, b^{t+1}).$$

Proof: Take $f(x) = \frac{1}{x^{t+1}}$, for which $f^{(4)}(x) = \frac{(t+1)(t+2)(t+3)(t+4)}{x^{t+5}} > 0$, for all $x \in (a,b)$, since $t > 0$.

and $f^{(2)}(x) = \frac{(t+1)(t+2)}{x^{t+3}} > 0$, for all $x \in (a,b)$, hence $f(x)$ is convex function.

Thus, for $f(x) = \frac{1}{x^{t+1}}$, the equations (2.1) and (2.3) together expressed as;

$$\frac{1}{b-a}.\frac{b^t - a^t}{a^t b^t} \leq \frac{1}{6}\left[\frac{1}{a^{t+1}} + \frac{4}{(\frac{a+b}{2})^{t+1}} + \frac{1}{b^{t+1}}\right] \leq \frac{1}{2}\left[\frac{1}{a^{t+1}} + \frac{1}{b^{t+1}}\right]$$

on simplifying leads to,

$$G^2(a,b)\frac{b^t - a^t}{t(b-a)} \leq \frac{1}{3}\left[A(a^{t+1}, b^{t+1}) + 2\frac{G^{2t+2}(a,b)}{A^{t+1}(a,b))}\right] \leq A(a^{t+1}, b^{t+1})$$

use the well known identity,

$$G^2(a,b) = A(a,b)H(a,b)$$

in the above equation leads to,

$$E_{s,1}^{t-1}(a,b) \leq \left[\frac{A(a^{t+1},b^{t+1}) + 2H^{t+1}(a,b)}{3G^2(a,b)}\right] \leq A(a^{t+1},b^{t+1}).$$

This completes the proof of Theorem 2.4.

Theorem: 2.5 *If $a, b > 0$, then the following inequality holds:*

$$E_{s,1}^{2t-1}(a,b) \leq \left[\frac{A(a^{2t-1},b^{2t-1}) + 2A^{2t-1}(a,b)}{3}\right] \leq A(a^{2t-1},b^{2t-1}).$$

Proof: Take $f(x) = x^{2t-1}$, for which $f^{(4)}(x) = (2t-1)(2t-2)(2t-3)(2t-4)x^{2t-5} > 0$, for all $t \in (-\infty, \frac{1}{2}] \cup [1, \frac{3}{2}] \cup [1, \infty)$, and $f^{(2)}(x) = (2t-1)(2t-2)x^{2t-3} > 0$, for all $t \in (-\infty, \frac{1}{2}] \cup [1, \frac{3}{2}] \cup [1, \infty)$, hence $f(x)$ is convex function.

Thus the proof of Theorem 2.5 follows for $f(x) = x^{2t-1}$.

References

[1] Bullen, P. S. *Handbook of means and their inequalities*, Kluwer Acad. Publ., Dordrecht, 2003.

[2] Hardy, G. H., J. E. Littlewood, G. Pòlya, *Inequalities*, 2nd edition, Cambridge University Press, Cambridge, 1959.

[3] Padmanabhan, S., V. Lokesha, M. Saraj and K. M. Nagaraja, Oscillatory mean for several positive arguments, *Journal of intelligent system research*, Vol. 2, 2008, No. 2, 137–139.

[4] Czinder, P., Z. Pales, An extension of the Hermite-Hadamard inequality and an application for Gini and Stolarsky means, *JIPAM*, Vol. 5, 2004, No. 2, Article No. 42.

[5] Sándor, J., M. Bencze, An application of Gauss Quadrature Formula, *Octogon. Mathematical Magazine*, Vol. 15, 2007, No. 1, 276–279.

[6] Wang, S., Y. Chu, The best bounds of combination of arithmetic and harmonic means for the Seiffert's mean, *Int. J. Math. Analysis*, Vol. 4, 2010, No. 22, 1079–1084.