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The structure of 'Pi'

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Abstract: Classes of the modular ring Z_4 were substituted into convergent infinite series for π and $\sqrt{2}$ to obtain Q, the ratio of the arc of a circle to the side of an inscribed square to yield $\pi = 2\sqrt{2}\,Q$. The corresponding convergents of the continued fractions for π, $\sqrt{2}$ and Q were then considered, together with the class patterns of the modular rings $\{Z_4, Z_5, Z_6\}$ and decimal patterns for π.

Keywords: Integer structure analysis, Modular rings, Prime numbers, Fibonacci numbers, Arctangents, Infinite series, Pell sequence, Continued fractions, Triangular numbers.

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1 Introduction

The number 'Pi', π, has been studied since antiquity [1, 13]. In particular, various mathematicians have developed calculations around inverse tangents and power series: John Wallis (1616–1703) [see Equation (1.1)], James Gregory (1638–1675) [14], Gottfried Wilfred Leibnitz (1646–1716) [13], John Machin (1680–1751) [17], Leonard Euler (1701–1783) [2], Robert Simson (1687–1768) [3], Carl Friedrich Gauss (1777–1855) [15], to mention but a few famous names from the history of mathematics.

With the dawn of differential calculus, the Greek method of inscribed and circumscribed polygons was replaced by convergent infinite series and algebraic and trigonometric methods. Wallis' elegant formula was

$$\frac{\pi}{2} = \frac{2}{1} \times \frac{2}{3} \times \frac{4}{3} \times \frac{4}{5} \times \frac{6}{5} \times \frac{6}{7} \times \frac{8}{7} \times \frac{8}{9} \times \ldots \tag{1.1}$$

Johan Heinrich Lambert (1728–1777) was the first to provide a rigorous proof that π is incommensurable [5]: π cannot be the root of a rational algebraic equation. The advent of electronic computers has since revived interest in a variety of techniques which have also enriched pure mathematics [8,17]: Google now has more than 1 million places listed! What more can be said? Here we outline how Integer Structure Analysis (ISA) permits a different analysis of the infinite series for π through functions of the rows of modular rings when detailed as modular arrays as in Table 1.

Row	$f(r)$	$4r_0$	$4r_1+1$	$4r_2+2$	$4r_3+3$
	Class	$\bar{0}_4$	$\bar{1}_4$	$\bar{2}_4$	$\bar{3}_4$
0		0	1	2	3
1		4	5	6	7
2		8	9	10	11
3		12	13	14	15
4		16	17	18	19
5		20	21	22	23
6		24	25	26	27
7		28	29	30	31

Table 1. Rows of Z_4

2 Ratio of arc of a quadrant of a circle to the side of an inscribed square

In a circle of radius r, the arc length of a quadrant is $\frac{1}{2}\pi r$, and the length of the side of an inscribed square is $\sqrt{2}r$, so that the ratio of the arc of length of the quadrant to the side of the inscribed square is $\pi/2\sqrt{2}$.

We now use the modular ring Z_4 to convert Equation (1.1) to

$$\frac{\pi}{2}=\prod_{r=0}^{\infty}\frac{(4r+2)^2(4r+4)^2}{(4r+1)(4r+3)^2(4r+5)} \tag{2.1}$$

combining segments of four fractions for each r, and

$$\sqrt{2}=\prod_{r=0}^{\infty}\frac{(4r+2)^2}{(4r+2)^2-1} \tag{2.2}$$

so that from (2.1) and (2.2) we have that

$$\begin{aligned}\frac{\pi}{2\sqrt{2}}&=\prod_{r=0}^{\infty}\frac{(4r+4)^2}{(4r+4)^2-1}\\&=Q.\end{aligned} \tag{2.3}$$

If we use the values of π and $\sqrt{2}$ to 30 decimal places, we arrive at

$$Q = 1.110720734882156630963 7,$$

and when r = 175, Equation (2.3) yields

$$Q = 1.1107212.$$

The series for Q and $\sqrt{2}$ are similar in format: the product of even and odd fractions. The numerator, $4r + 2$, for $\sqrt{2}$ is in Class $\bar{2}_4$. Thus, the even numerator of the multiple fractions produced will contain odd factors for all values of r, whereas the variable $(4r + 4) \in \bar{0}_4$ will not, and those factors which do occur will cancel out. So the structures of the product fractions are

$$\frac{2^n p_1 p_2 \cdots}{p_1' p_2'} \text{ for } \sqrt{2}$$

and

$$\frac{2^m}{p_1 p_2 \cdots} \text{ for } Q.$$

The structure of π is thus resolved into

$$\pi = 2\sqrt{2}Q. \tag{2.4}$$

Q is irrational but can be useful in the analysis of the decimal expansion of π obtained from Q and $2\sqrt{2}$. Moreover, the “factorisation” of π should be useful in the context of continued fractions as we now illustrate.

Every positive number can be expressed uniquely as a regular continued fraction [5].

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}} = a_0 + \frac{1}{a_1 +}\frac{1}{a_2 +}\frac{1}{a_3 +}\cdots$$
$$= [a_0; a_1, a_2, a_3, \dots]. \tag{2.5}$$

The partial quotients $\{a_1, a_2, a_3, \dots\}$ for $\sqrt{2}$ have a simple pattern

$$\sqrt{2} = [1; 2,2,2,2,2,\dots],$$

but no such pattern has been found for π [5].

Convergents have remarkable properties and they are obtained by stopping the continued fractions at each successive stage [12]. The first six convergents for $\sqrt{2}$ and π may be used to calculate those for Q (Table 2).

Number	**Convergents**					
$\sqrt{2}$	$\frac{1}{1}$	$\frac{3}{2}$	$\frac{7}{5}$	$\frac{17}{12}$	$\frac{41}{29}$	$\frac{99}{70}$
π	$\frac{3}{1}$	$\frac{22}{7}$	$\frac{333}{106}$	$\frac{355}{113}$	$\frac{103993}{33102}$	$\frac{104348}{33215}$
Q	$\frac{3}{2}$	$\frac{22}{21}$	$\frac{1665}{1484}$	$\frac{2130}{1921}$	$\frac{3015797}{2714364}$	$\frac{104348}{93951}$

Table 2. First six convergents for $\sqrt{2}$, π and Q

The convergents $\frac{b}{c}$ satisfy the second order linear recurrence relations [13]:

$$b_{n+1} = a_n b_n + b_{n-1},\ c_{n+1} = a_n c_n + c_{n-1}. \tag{2.6}$$

with suitable initial conditions. Thus, in the first row of Table 2, the numerators and denominators of the convergent to $\sqrt{2}$ satisfy the Pell recurrence relation in which $a_n = 2$, with initial terms 1 and 3 in the numerators $\{b_n\}$, and 1 and 2 in the denominators $\{c_n\}$ (the standard Pell sequence [4]). There too $c_{n-1} + c_n = b_n$, analogous to the identity which relates the Fibonacci and Lucas numbers. We further observe that if $u_n = b^2 c^2$ where $\frac{b}{c}$ is any convergent to the continued fraction for $\sqrt{2}$, then the elements of

$$\{u_n\} = \{1,36,1225,41616,1413721,48024900,..\}$$

are square-triangular numbers [6] with the formal generating function

$$\sum_{n=0}^{\infty} u_n x^n = \frac{1+x}{1-x+x^2-x^3}. \tag{2.7}$$

Questions about square-pentagonal and triangular-pentagonal numbers are still open [15]. Finally, we note that $Q = \frac{1}{2}\left(\sqrt{1.4\dot{9}\dot{1}}+1\right)$, in which the argument of the radical is rational.

3 Distribution of decimal places of *π*

3.1 Modular rings

The consecutive positions of the decimals of π do no appear to have a regular pattern, but we can analyse these positions indirectly. We do these by classifying them in terms of their position, *n*, in the array of a modular ring. See Tables 1,3,4 for Z_4, Z_5 and Z_6. See also [9,10,11].

Row	*f*(*r*)	$5r_0$	$5r_1+1$	$5r_2+2$	$5r_3+3$	$5r_4+4$
	Class	$\bar{0}_5$	$\bar{1}_5$	$\bar{2}_5$	$\bar{3}_5$	$\bar{4}_5$
0		0	1	2	3	4
1		5	6	7	8	9
2		10	11	12	13	14
3		15	16	17	18	19
4		20	21	22	23	24
5		25	26	27	28	29

Table 3. Rows of Z_5

Row	*f*(*r*)	$6r_1-2$	$6r_2-1$	$6r_3$	$6r_4+1$	$6r_5+2$	$6r_6+3$
	Class	$\bar{1}_6$	$\bar{2}_6$	$\bar{3}_6$	$\bar{4}_6$	$\bar{5}_6$	$\bar{6}_6$
0		-2	-1	0	1	2	3
1		4	5	6	7	8	9
2		10	11	12	13	14	15
3		16	17	18	19	20	21
4		22	23	24	25	26	27
5		28	29	30	31	32	33

Table 4. Rows of Z_6

The sequence of classes provides patterns that have some repetitions for the 100 decimal places considered (Table 5), but do these patterns persist for the more than billion places already known?

	o e	Total	Z_4	Z_5	Z_6
1	5 3	8	13301023	13204340	46414512
2	8 4	12	201011310311	111333331343	316162641226
3	7 4	11	13101332023	40240231411	66234241154
4	5 5	10	2330130230	2431240022	5423623165
5	3 5	8	00230312	43013110	15143643
6	4 5	9	302110322	202142023	451263615
7	5 3	8	11330203	34421114	42625336
8	4 8	12	322230322100	131402243143	235121453631
9	4 10	14	10222201322300	02403240032400	23535356415451
0	5 3	8	022113111	20401202	55322244

Table 5: Class patterns for *n* (position of number in decimal array for π)
Legend: o=odd; e=even; bars ans subscripts are omitted for notational brevity in the elements of Z_4,Z_5,Z_6: e.g., $\bar{3}_4$ is represented by 3, etc.

3.2 Decimal patterns

If each 100 decimals are placed in a 10×10 array, the column, row and number *n* of the position of the decimal in the array will be characteristic of that number. Comparison of sequential 100 decimals can then be made. *n**, the right-end-digit (RED) of *n*, will equal the column in which the number falls. For example, if *n* = 9, 19, 29, 39,…, the number will fall in column 9 (Table 6).

When 300 decimal places are considered (three arrays), each decimal number has characteristic features in terms of rows in which they do not occur (Table 7). Also the appearances of these numbers in the sequence, *n*, (1^{st} decimal *n* = 1) have certain REDs that do not occur. For instance, for 7, *n* never has a RED of 1, while 3 never has a RED of 8. Thus for the first 300 decimals, 7 never occurs for *n* = 1, 11, 21, 31, …, 291, while 3 never occurs for *n* = 8, 18, 28, 38, …, 298.

N = 2			N = 4			N = 8		
Col	**Row**	***n***	**Col**	**Row**	***n***	**Col**	**Row**	***n***
6	1	6	2	1	2	1	2	11
6	2	16	9	2	19	8	2	18
1	3	21	3	3	23	6	3	26
8	3	28	6	4	36	4	4	34
3	4	33	7	6	57	5	4	35
3	6	53	9	6	59	2	6	52
3	7	63	10	6	60	7	7	67
3	8	73	10	7	70	4	8	74
6	8	76	7	9	87	8	8	78
3	9	83	2	10	92	1	9	81
3	10	93				4	9	84
						8	9	88

Table 6(a): $N = 2^n$

N = 3			N = 6			N = 9		
Col	**Row**	***n***	**Col**	**Row**	***n***	**Col**	**Row**	***n***
9	1	9	7	1	7	5	1	5
5	2	15	10	2	20	2	2	12
7	2	17	2	3	22	4	2	14
4	3	24	1	5	41	10	3	30
5	3	25	9	7	69	8	4	38
7	3	27	2	8	72	2	5	42
3	5	43	5	8	75	4	5	44
6	5	46	2	9	82	5	5	45
4	7	64	8	10	98	5	6	55
6	9	86				8	6	58
1	10	91				2	7	62
						9	8	79
						10	8	80
						10	10	100

Table 6(b). 3|*N*

N = 0			N = 1			N = 5			N = 7		
Col	**Row**	***n***	**Col**	**Row**	***n***	**Col**	**Row**	***n***	**Col**	**Row**	***n***
22	4	32	1	1	1	4	1	4	3	2	13
10	5	50	3	1	3	8	1	8	9	3	29
4	6	54	7	4	37	10	1	10	9	4	39
5	7	65	10	4	40	1	4	31	7	5	47
1	8	71	9	5	49	8	5	48	6	6	56
7	8	77	8	7	68	1	6	51	6	7	66
5	9	85	4	10	94	1	7	61	6	10	96
7	10	97				10	9	90			

Table 6(c). $N < 9$

Rows missing					***n** missing**			
Decimal Numbers	**1st 100**	**2nd 100**	**3rd 100**	**1–300**	**1st 100**	**2nd 100**	**3rd 100**	**1–300**
0	1,2,3	9	2,3,4,6,8	---	3,6,8,9	0,3	1,2,3,6,9	3
1	2,3,6,8,9	2,3,9	3,6,8	3	2,5,6	1,2,6,7,9	4,2	2
2	5	3,10	2,4,7	---	0,2,4,5,7,9	1,7,8	2,6,7	7
3	4,6,8	1,6,8,9	1,5,6	6	0,2,8	0,2,8,9	8,9	8
4	5,8	4	4,5,9	---	1,4,5,8	0	5,9	---
5	2,3,8,10	2,7,9	5,7,9,10	---	2,3,5,6,7,9	4,5,6	1,4,7,8,9	---
6	4,6	4,5,6,7,8	1,5,10	---	3,4,6	2,3,5,6,9	5	---
7	1,8,9	1,3,5,8,9,10	2,6,7,8	8	0,1,2,4,5,8	1,3,5,8,9	1,3,6,7,10	1
8	1,5,10	---	2,5,9,10	---	3,9	6,8	1,10	---
9	9	1,2,4,6	3,4,7,8	---	1,3,6,7	1,5,6,7,8	1,2,3,5,6,10	1,6

Table 7. Distribution of decimals n = 1–300.

4 Final Comments

Billions of decimal places have been calculated for π and it continues to be the subject of many papers. REDs and ISA with modular rings introduce some new perspectives.

Finally, it is of interest to note that from the work of the famous computational mathematician, Derek Lehmer [7], came the neat and relevant result that the arctangents of the reciprocals of alternate odd-indexed Fibonacci numbers sum to π/4.

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