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On some identities of Ramanujan–Göllnitz–Gordon continued fraction

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Abstract: In this paper, we give an alternative and simple proof of certain identities of Ramanujan–Göllnitz–Gordon continued fraction.
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1 Introduction

Throughout the paper, we assume $|q| < 1$ and for positive integer n, we use the standard notation

$$(a;q)_n = \prod_{k=0}^{n-1}(1 - aq^k)$$

and

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1 - aq^k).$$

In his notebook [10], [3, p. 34], Ramanujan defines his general theta function $f(a,b)$ by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \quad |ab| < 1.$$

Following Ramanujan, we define

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty},$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty.$$

For convenience, we denote $f(-q^n) := f_n$

Let,

$$H(q) := \frac{q^{1/2}}{1+q}{}_{+}\frac{q^2}{1+q^3}{}_{+}\frac{q^4}{1+q^5}{}_{+}\frac{q^6}{1+q^7}{}_{+\ldots} \tag{1.1}$$

denote the Ramanujan–Göllnitz–Gordon continued fraction. On page 229 of his second notebook[10], Ramanujan recorded a product representation of $H(q)$, namely

$$H(q) = q^{1/2}\frac{(q;q^8)_\infty(q^7;q^8)_\infty}{(q^3;q^8)_\infty(q^5;q^8)_\infty}, \tag{1.2}$$

along with the following two identities for $H(q)$:

$$\frac{1}{H(q)} - H(q) = \frac{\varphi(q^2)}{q^{1/2}\psi(q^4)} \tag{1.3}$$

and

$$\frac{1}{H(q)} + H(q) = \frac{\varphi(q)}{q^{1/2}\psi(q^4)}. \tag{1.4}$$

Without any knowledge of Ramanujan's work, Göllnitz [8] and Gordon [7] rediscovered and proved (1.2) independently. Later G. E. Andrews [1] proved (1.2) as a corollary of a more general result.

The Göllnitz–Gordon functions are defined as

$$S(q) := \sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^2;q^2)_n} q^{n^2} = \frac{1}{(q;q^8)_\infty(q^4;q^8)_\infty(q^7;q^8)_\infty}$$

and

$$T(q) := \sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^2;q^2)_n} q^{n^2+2n} = \frac{1}{(q^3;q^8)_\infty(q^4;q^8)_\infty(q^5;q^8)_\infty}.$$

We note that

$$H(q) := q^{1/2}\frac{T(q)}{S(q)}. \tag{1.5}$$

H. H. Chan and S. S. Haung [4], gave many new identities involving $H(q)$, including relations between $H(q)$ and $H(q^3)$, $H(q)$ and $H(q^4)$. K. R. Vasuki and B. R. Srivatsa Kumar [12] obtained new identities relating $H(q)$ with $H(q^5)$, $H(q^7)$ and $H(q^{11})$ by employing modular equations given by Ramanujan. Recently B. Cho, J. K. Koo and Y. K. Park [6] have extended the above results on the continued fraction $H(q)$ to all odd primes p by computing the affine models of modular curves $X(\Gamma)$ with Γ=$\Gamma_1(8) \bigcap \Gamma_0(16p)$.

S. S. Huang [9] established many modular relation involving Göllnitz–Gordon fuctions which are analogous to the famous Ramanujan's fourty identities. He further extracted interesting partition results from some of the modular relations. Later S. L. Chen and Huang [5], N. D. Baruah, J. Bora and N. Saikia [2] further established new modular relations for the Göllnitz- Gordon functions.

Employing the modular relations found in [9], in Section 3 of this paper we established an alternate proof of relations between $H(q)$ and $H(q^n)$, n = 2, 3, 5 and 7. In Section 2, we recall some known results, which will be used to prove identities in Section 3.

2 Some preliminary results

Lemma 2.1. We have

$$\varphi(q) = \frac{{f_2}^5}{{f_1}^2{f_4}^2}, \qquad \varphi(-q) = \frac{{f_1}^2}{f_2}, \qquad \psi(q) = \frac{{f_2}^2}{f_1}, \qquad f(q) = \frac{{f_2}^3}{f_1 f_4}$$

$$\psi(-q) = \frac{f_1 f_4}{f_2}, \qquad \chi(q) = \frac{f_2^2}{f_1 f_4} \qquad \chi(-q) = \frac{f_1}{f_2}, \qquad S(q)T(q) = \frac{f_2 f_8^2}{f_1 f_4^2}$$

$$S(q) = \frac{f_8^2}{f_4 f(-q,-q^7)} \qquad \text{and} \qquad T(q) = \frac{f_8^2}{f_4 f(-q^3,-q^3)}.$$

For a proof of the above identities, one may refer [9] and [2].

Lemma 2.2. [4]. We have

$$\frac{\varphi(q^2)}{\varphi(q)} = \frac{1-H^2(q)}{1+H^2(q)}.$$

Lemma 2.3. We have

$$q^{1/2}\left(\frac{f_4}{f_1}\right)^4 = \frac{H(q)[H^2(q)-1]}{1+H^4(q)-6H^2(q)}.$$

Proof. From Entry 25 (vii) [3, p. 40], we have

$$\varphi^4(q) - \varphi^4(-q) = 16q\psi^4(q^2)$$

which can be rewritten as

$$\frac{\varphi^4(q)}{\varphi^4(-q)} - 1 = 16q\frac{\psi(q^2)}{\varphi^4(-q)}. \tag{2.1}$$

From Lemma 2.1., we have

$$\frac{\psi(q^2)}{\varphi(-q)} - 1 = \left(\frac{f_4}{f_1}\right)^4. \tag{2.2}$$

Substituting (2.2) in (2.1), we see that

$$\frac{\varphi^4(q)}{\varphi^4(-q)} - 1 = 16q\left(\frac{f_4}{f_1}\right)^8. \tag{2.3}$$

Now from Entry 25 (vi) [3, p. 40], we have

$$\varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2),$$

which can be rewritten as

$$\frac{\varphi^2(-q)}{\varphi^2(q)} = 2\frac{\varphi^2(q^2)}{\varphi^2(q)} - 1.$$

Using the above identity on the left hand side of (2.3), we obtain

$$16q\left(\frac{f_4}{f_1}\right)^8 = \frac{1}{\left[2\frac{\varphi^2(q^2)}{\varphi^2(q)} - 1\right]}.$$

Employing Lemma 2.2. on the righthand side of the above and then after some simplification yields the required result.

Lemma 2.4. [9] We have

$$S(q^4)T(q^2) - qS(q^2)T(q^4) = \frac{f_1 f_{32}}{f_2 f_{16}}.$$

Lemma 2.5. [13] If $P_n := \dfrac{\varphi(q^n)}{\varphi(-q^n)}$ and $Q_n := \dfrac{\varphi^2(q^{2n})}{\varphi^2(q^n)}$, then

$$2P_2^2 = P_1 + \frac{1}{P_1} \tag{2.4}$$

and

$$\frac{1}{P_n^2} + 1 = 2Q_n. \tag{2.5}$$

Lemma 2.6. [9] We have

$$S(q^3)S(q) + qT(q^3)T(q) = \frac{f_3 f_4}{f_1 f_{12}}$$

and

$$S(q^3)T(q) - qS(q)T(q^3) = \frac{f_1 f_{12}}{f_3 f_4}.$$

Lemma 2.7. [9] We have

$$S(q^5)S(q) + q^3T(q^5)T(q) = \frac{f_2 f_{10}}{f_1 f_{20}}$$

and

$$S(q^5)T(q) - q^2S(q)T(q^5) = \frac{f_2 f_{10}}{f_4 f_5}.$$

Lemma 2.8. [3, p. 69] If μ is even, then

$$\psi(q^{\mu+\nu})\psi(q^{\mu-\nu}) = \varphi(q^{\mu(\mu^2-\nu^2})\psi(q^{2\mu}) + \sum_{m=1}^{(\mu/2)-1} q^{\mu m^2-\nu m} f(q^{(\mu+2m)(\mu^2-\nu^2)}, q^{(\mu-2m)(\mu^2-\nu^2)})$$

$$f(q^{2\nu m}, q^{2\mu-2\nu m}) + q^{(\mu^3/4)-(\mu\nu/2)}\psi(q^{2\mu(\mu^2-\nu^2)} f(q^{\mu\nu}, q^{2\mu-\mu\nu}).$$

Lemma 2.9. We have

$$\left(\frac{\psi(-q)}{\psi(q)}\right)^2 = \frac{\varphi(-q)\varphi(q^2)}{\psi(q^2)\psi(q^4)}.$$

Proof. From Entry 25 (iv) [3, p. 40], we have

$$\psi^2(q) = \varphi(q)\psi(q^2). \tag{2.6}$$

Changing q to $-q$ and then dividing throughout by $\psi^2(q^4)$ in the above identity, we obtain

$$\frac{\psi^2(-q)}{\psi^2(q^4)} = \frac{\varphi(-q)\psi(q^2)}{\psi^2(q^4)}.$$

Changing q to q^2 in (2.6) and employing the same in the righthand side of the above identity, we obtain the required result.

Lemma 2.10. [9] We have

$$S(q^7)T(q) - q^3S(q)T(q^7) = 1.$$

3 Main results

In this section, we give alternating proof of continued fraction $H(q)$ with the continued fractions $H(q^n)$ for n= 2, 3, 5 and 7.

Theorem 3.1. [4]. Let $u = H(q)$ and $v = H(q^2)$. Then,

$$u^2 = v\frac{1-v}{1+v}.$$

Proof. Let $x = H(q^2)$ and $y = H(q^4)$. By Lemma 2.4., we have

$$S(q^4)T(q^2) - qS(q^2)T(q^4) = \frac{f_1f_{32}}{f_2f_{16}}. \tag{3.1}$$

Changing q to $-q$ in (3.1), we obtain

$$S(q^4)T(q^2) + qS(q^2)T(q^4) = \frac{f_2^2f_{32}}{f_1f_4f_{16}}. \tag{3.2}$$

Dividing (3.1) by (3.2) and using Lemma 2.1., we have

$$\frac{S(q^4)T(q^2) - qS(q^2)T(q^4)}{S(q^4)T(q^2) + qS(q^2)T(q^4)} = \frac{\varphi(-q)}{\varphi(-q^2)}.$$

Now using (1.5) on the left hand side of the above identity, we obtain

$$\frac{H(q^2)-H(q^4)}{H(q^2)+H(q^4)} = \frac{\varphi(-q)}{\varphi(-q^2)}.$$

Then, by Lemma 2.5., we can write

$$\frac{1}{P_1} = \left(\frac{H(q^2)-H(q^4)}{H(q^2)+H(q^4)}\right)^2 \tag{3.3}$$

Setting $n = 2$ in (2.5) and then employing (2.4), we find that

$$\frac{2P_1}{P_1^2+1} = 2Q_2 - 1 \tag{3.4}$$

Changing q to q^2 in Lemma 2.1, we observe that

$$Q_2 = \frac{\varphi(q^4)}{\varphi(q^2)} = \frac{1-H^2(q^2)}{1+H^2(q^2)}.$$

Using (3.3) and the above identity in (3.4) and then factorizing, we find that

$$(y^2 - y + yx^2 + x^2)(-y^2 - y + yx^2 - x^2) = 0 \tag{3.5}$$

Now by definition of $H(q)$, we have

$$x = q(1 - q^2 + q^6 - q^8 + q^{10} - 2q^{14} + 2q^{16} - q^{18} + 2q^{22} - 4q^{24} + 3q^{26} - 3q^{30} + \cdots) \tag{3.6}$$

and

$$y = q^2(1 - q^4 + q^{12} - q^{16} + q^{20} - 2q^{28} + 2q^{32} - q^{36} + 2q^{44} - 4q^{48} + 3q^{52} - 3q^{60} \cdots) \tag{3.7}$$

Using (3.6) and (3.7) in (3.5), we see that the first and second factor becomes

$$-q^{26}(2 - 2q^4 + 4q^6 + 10q^8 - 12q^{10} - 8q^{12} + 18q^{14} - 2q^{18} + \cdots)$$

and

$$-q^2(2 - 2q^2 + 2q^4 - 8q^8 + 10q^{10} - 2q^{12} - 8q^{14} + 22q^{16} + \cdots),$$

which implies that second factor does not vanish. Hence,

$$y^2 - y + yx^2 + x^2 = 0.$$

Now changing q to $q^{1/2}$ in the above, we get the required result.

Theorem 3.2. [12] Let $u = H(q)$ and $v = H(q^3)$. Then,

$$u^4v^3 - 3u^3v^2 + u^3 - 3u^2v^3 + 3u^2v - uv^4 + 3uv^2 - v = 0.$$

Proof. From Lemma 2.6., we have

$$\frac{S(q^3)T(q) - qS(q)T(q^3)}{S(q^3)S(q) + qT(q^3)T(q)} = \frac{f_1^2 f_{12}^2}{f_3^2 f_4^2}.$$

Multiplying $q^{1/2}$ on both sides and then employing (1.5) on the left hand side of above, we obtain

$$\frac{u - v}{1 + uv} = q^{1/2} \left(\frac{f_1}{f_4}\right)^2 \left(\frac{f_{12}}{f_3}\right)^2.$$

Squaring both sides and then employing Lemma 2.3. on the right hand side of the above identity, we find that

$$\left(\frac{u - v}{1 + uv}\right)^2 = \left(\frac{1 + u^4 - 6u^2}{u(u^2 - 1)}\right) \left(\frac{v(v^2 - 1)}{1 + v^2 - 6v^2}\right).$$

Factorizing the above identity using Maple, we see that

$$(uv - v + 1 + u)(uv + v + 1 - u)(u^4v^3 - 3u^3v^2 + u^3 - 3u^2v^3$$
$$+3u^2v - uv^4 + 3uv^2 - v) = 0. \tag{3.8}$$

Now by definition of u and v, we have

$$u = q^{1/2}(1 - q + q^3 - q^4 + q^5 - 2q^7 + 2q^8 - q^9 + 2q^{11} - 4q^{12} + \cdots) \tag{3.9}$$

and

$$v = q^{3/2}(1 - q^3 + q^9 - q^{12} + q^{15} - 2q^{21} + 2q^{24} - q^{27} + 2q^{33} - 4q^{36} + \cdots). \tag{3.10}$$

Using (3.9) and (3.10) in (3.8), we see that the first, second and the third factor becomes,

$$1 + q^{1/2}(1 - 2q + q^{3/2} - q^{5/2} + q^3 + q^5 + q^{13/2} - 2q^7 - q^{15/2} - q^{17/2} - q^{10} + \cdots),$$

$$1 - q^{1/2}(1 - 2q - q^{3/2} + q^{5/2} + q^3 + q^5 - q^{13/2} - 2q^7 - q^{15/2} - 2q^8 - q^{17/2} + q^9 + \cdots).$$

and

$$q^{27/2}(3 - 3q + 15q^4 - 6q^5 - 18q^6 + 9q^7 + 42q^9 + 51q^{12} + 6q^{13} + 60q^{14} - 156q^{15} + \cdots),$$

which implies that the first and second factor does not vanish. Hence,

$$u^4v^3 - 3u^3v^2 + u^3 - 3u^2v^3 + 3u^2v - uv^4 + 3uv^2 - v = 0$$

This completes the proof.

Theorem 3.3. [12] Let $u = H(q)$ and $v = H(q^5)$. Then,

$$u^6v^5 - 5u^5v^2 + u^5 - 5u^4v^5 + 10u^4v^3 - 10v^4u^3 + 10u^3v^2 - 10u^2v^3 + 5u^2v$$

$-uv^6 + 5uv^4 - v = 0.$

Proof. From Lemma 2.7., we have

$$\frac{S(q^5)T(q) - q^2S(q)T(q^5)}{S(q^5)S(q) + q^3T(q^5)T(q} = \frac{f_1f_{20}}{f_4f_5}.$$

Multiplying the above identity throughout by $q^{1/2}$ and using (1.5) on the left hand side, we obtain

$$\frac{u-v}{1+uv} = q^{1/2}\frac{f_1f_{20}}{f_4f_5}.$$

Taking power 4 on both sides of the above and employing Lemma 2.3. on the righthand side and then factorizing using Maple, we deduce that

$$(uv - v + 1 + u)(uv + v + 1 - u)(u^6v^5 - 5u^5v^2 + u^5 - 5u^4v^5 + 10u^4v^3$$

$$-10u^3v^4 + 10u^3v^2 - 10u^2v^3 + 5u^2v - uv^6 + 5uv^4 - v) = 0 \tag{3.11}$$

Now, from the definition of u and v, we have

$$u = H(q) = q^{1/2}(1 - q + q^3 - q^4 + q^5 - 2q^7 + 2q^8 - q^9 + 2q^{11} - \cdots) \tag{3.12}$$

and

$$v = H(q^5) = q^{5/2}(1 - q^5 + q^{15} - q^{20} + q^{25} - 2q^{35} + 2q^{40} - q^{45} + 2q^{55} - \cdots). \tag{3.13}$$

Using (3.12) and (3.13) in (3.11), we see that the first, second and the third factor becomes,

$$1 + q^{1/2}(1 - q - q^2 + q^{5/2} + q^3 - q^{7/2} - q^4 + q^5 + q^{11/2} - q^{13/2} - q^7 + 2q^8 + q^{17/2} + \cdots),$$

$$1 - q^{1/2}(1 - q - q^2 - q^{5/2} + q^3 + q^{7/2} - q^4 - q^5 - q^{11/2} + q^{13/2} - q^7 + 2q^8 - q^{17/2} + \cdots),$$

and

$$-q^{29/2}(5+30q^2-15q^3-25q^4+40q^5+75q^6-195q^7+60q^8+335q^9-565q^{10}+25q^{11}+995q^{12}+\cdots),$$

respectively. Then, it clearly follows that the first and the second factor do not vanish. Hence,

$$u^6v^5 - 5u^5v^2 + u^5 - 5u^4v^5 + 10u^4v^3 - 10v^4u^3 + 10u^3v^2 - 10u^2v^3 + 5u^2v$$
$$-uv^6 + 5uv^4 - v = 0$$

This completes the proof.

Theorem 3.4. [12] Let $u := H(q)$ and $v := H(q^7)$. Then,

$$7u^3v + v^8 + u^8 - uv - 49u^3v^3 - 7u^5v + 7uv^3 - 7uv^5 - 7uv^7 + 28u^2v^6 - 7u^3v^5$$

$$+70u^4v^4 - 7u^5v^3 + 28u^6v^2 - 7u^7v - 7u^3v^7 + 7u^7v^5 - 7u^7v^3 + 7u^5v^7 - 49u^5v^5 - u^7v^7 = 0.$$

Proof. Putting $\mu = 4$, $\nu = 3$ in Lemma 2.8., it can be shown [3, p. 315] that

$$\psi(q)\psi(q^7) = \varphi(q^{28})\psi(q^8) + q\psi(q^{14})\psi(q^2) + q^6\psi(q^{56})\varphi(q^4).$$

Changing q to $-q$ in the above and then adding the resultant identity with the above identity, we find that

$$\psi(q)\psi(q^7) + \psi(-q)\psi(-q^7) = 2\varphi(q^{28})\psi(q^8) + 2q^6\psi(q^{56})\varphi(q^4). \tag{3.14}$$

From Corollary [3, p. 40] and Entry 31 [3, p. 48], we have

$$\psi(q) = f(q^6, q^{10}) + qf(q^2, q^{14}).$$

Changing q to q^7 in the above identity and then multiplying the resultant identity with the above, we see that

$$\psi(q)\psi(q^7) = f(q^6, q^{10})f(q^{42}, q^{70}) + qf(q^2, q^{14})f(q^{42}, q^{70}) +$$
$$q^7f(q^6, q^{10})f(q^{14}, q^{98}) + q^8f(q^2, q^{14})f(q^{14}, q^{98}).$$

Changing q to $-q$ in the above and then adding the resultant identity with the above identity, we obtain

$$\psi(q)\psi(q^7) + \psi(-q)\psi(-q^7) = 2f(q^6, q^{10})f(q^{42}, q^{70}) + 2q^8f(q^2, q^{14})f(q^{14}, q^{98}).$$

Now from (3.14) and (3.15), we have

$$f(q^6, q^{10})f(q^{42}, q^{70}) + q^8f(q^2, q^{14})f(q^{14}, q^{98}) = \varphi(q^{28})\psi(q^8) + q^6\psi(q^{56})\varphi(q^4).$$

Changing q to $q^{1/2}$ and then q to $-q$ in the above identity, we have

$$f(-q^3, -q^5)f(-q^{21}, -q^{35}) + q^4f(-q, -q^7)f(-q^7, -q^{49}) = \varphi(q^{14})\psi(q^4) - q^3\psi(q^{28})\varphi(q^2).$$

Using Lemma 2.1. on the left hand side of the above identity, we obtain

$$\frac{1}{T(q)T(q^7)} + q^4\frac{1}{S(q)S(q^7)} = \frac{f_4f_{28}}{f_8^2f_{56}^2}\left[\varphi(q^{14})\psi(q^4) - q^3\psi(q^{28})\varphi(q^2)\right].$$

Multiplying the above identity throughout by $S(q)S(q^7)T(q)T(q^7)$ and then using Lemma 2.1. on the righthand side, we obtain

$$S(q^7)T(q) + q^4S(q)T(q^7) = \frac{f_2f_{14}}{f_1f_4f_7f_{28}}\left[\varphi(q^{14})\psi(q^4) - q^3\psi(q^{28})\varphi(q^2)\right].$$

From Lemma 2.1. the righthand side of the above identity can be written as

$$S(q^7)T(q)+q^4S(q)T(q^7)=\left[\frac{\varphi(q^{14})\psi(q^4)}{\psi(-q)\psi(-q^7)}-q^3\frac{\psi(q^{28})\varphi(q^2)}{\psi(-q)\psi(-q^7)}\right]$$

The above identity can be rewritten as

$$S(q^7)T(q)+q^4S(q)T(q^7)=q^{7/2}\frac{\psi(q^4)\psi(q^{28})}{\psi(-q)\psi(-q^7)}\left[\frac{\varphi(q^{14})}{q^{7/2}\psi(q^{28})}-\frac{\varphi(q^2)}{q^{1/2}\psi(q^4)}\right].$$

Now from Lemma 2.10. and the above identity, we have

$$\frac{S(q^7)T(q)-q^3S(q)T(q^7)}{S(q^7)T(q)+q^4S(q)T(q^7)}=\frac{q^{-7/2}\dfrac{\psi(-q)}{\psi(q^4)}\dfrac{\psi(-q^7)}{\psi(q^{28})}}{\left[\dfrac{\varphi(q^{14})}{q^{7/2}\psi(q^{28})}-\dfrac{\varphi(q^2)}{q^{1/2}\psi(q^4)}\right]}.$$

Squaring on both sides of the above identity and then employing Lemma 2.9., we see that

$$\left(\frac{S(q^7)T(q)-q^3S(q)T(q^7)}{S(q^7)T(q)+q^4S(q)T(q^7)}\right)^2=\frac{q^{-7}\dfrac{\varphi(-q)}{\psi(q^2)}\dfrac{\varphi(q^2)}{\psi(q^4)}\dfrac{\varphi(-q^7)}{\psi(q^{14})}\dfrac{\varphi(q^{14})}{\psi(q^{28})}}{\left[\dfrac{\varphi(q^{14})}{q^{7/2}\psi(q^{28})}-\dfrac{\varphi(q^2)}{q^{1/2}\psi(q^4)}\right]^2}.$$

Now from (2.2) we have $\dfrac{\varphi(-q)}{\psi(q^2)}=\dfrac{f_1^2}{f_4^2}$.

Squaring on both sides of the above identity, then employing (1.5) on the left hand side, employing Lemma 2.1. and Lemma 2.2. on the righthand side, we deduce that

$$\left(\frac{u-v}{1+uv}\right)^4=\frac{(1-u^4-6u^2)(1-u^2)(1+v^4-2v^2)(1-v^2)(uv)}{(u-v)^4(1+uv)^4}$$

Factorizing the above identity using Maple, we get the required result.

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