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Schur convexity of Gnan mean for two variables

V. Lokesha[1], K. M. Nagaraja[2], Naveen Kumar B.[3] and Y.-D. Wu[4]

[1] Department of Mathematics, Acharya institute of Technology, Bangalore-90, India
e-mail: `lokiv@yahoo.com`

[2] Department of Mathematics, Sri Krishna Institute of Technology, Bangalore-90, India
e-mail: `kmn_2406@yahoo.co.in`

[3] Department Of Mathematics, R.N.S.Institute of Technology, Bangalore-61, India
e-mail: `naveenbkoda@yahoo.co.in`

[4] Xinchang High School, Xinchang City, Zhejiang Province 312500, P. R. China
e-mail: `zjxcwyd@tom.com`

Abstract: In this paper, the convexity and Schur convexity of the Gnan mean and its dual form in two variables are discussed.
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1 Introduction

For positive numbers a, b, let

$$I = I(a,b) = \begin{cases} \exp\left[\dfrac{b\ln b - a\ln a}{b-a} - 1\right], & a < b; \\ a, & a = b; \end{cases} \tag{1.1}$$

$$L = L(a,b) = \begin{cases} \dfrac{a-b}{\ln a - \ln b}, & a \neq b; \\ a, & a = b; \end{cases} \tag{1.2}$$

$$H = H(a,b) = \frac{a + \sqrt{ab} + b}{3}. \tag{1.3}$$

These are respectively called the Identric, Logarithmic and Heron means.

In [3, 7, 8], V. Lokesha et al. studied extensively and obtained some remarkable results on the weighted Heron mean, the weighted Heron dual mean and the weighted product type means and its monotonicities.

In [5, 6], Zhang et al. gave the generalizations of Heron mean, similar product type means and their dual forms. For two variables, the above means as follows:

$$I(a,b;k)=\prod_{i=1}^{k}\left(\frac{(k+1-i)a+ib}{k+1}\right)^{\frac{1}{k}},\quad I^{*}(a,b;k)=\prod_{i=0}^{k}\left(\frac{(k-i)a+ib}{k}\right)^{\frac{1}{k+1}},\tag{1.4}$$

and

$$H(a,b;k)=\frac{1}{k+1}\sum_{i=0}^{k}a^{\frac{k-i}{k}}b^{\frac{i}{k}},\quad h(a,b;k)=\frac{1}{k}\sum_{i=1}^{k}a^{\frac{k+1-i}{k+1}}b^{\frac{i}{k+1}},\tag{1.5}$$

where k is a natural number. Authors proved that $H(a,b;k)$ and $I^{*}(a,b;k)$ are monotone decreasing functions and $h(a,b;k)$ and $I(a,b;k)$ are monotone increasing functions with k and also established the following limitations.

$$\lim_{k\to+\infty}I(a,b;k)=\lim_{k\to+\infty}I^{*}(a,b;k)=I(a,b),$$

and

$$\lim_{k\to+\infty}H(a,b;k)=\lim_{k\to+\infty}h(a,b;k)=L(a,b).$$

In [2], V. Lokesha et al. defined the Gnan mean and its dual form for two variables. Also, they obtained some interesting properties, monotonic results and its limitations.

The detailed discussion on convexity and Schur convexity can be found in [1, 4].

We recall the definitions which are essential to develop this paper.

Definition 1.1. *[2] Let $a\geqslant 0$ and $b\geqslant 0$, and k be a non-negative integer, α,β two real numbers. The Gnan mean $G(a,b;k,\alpha,\beta)$ and its dual $g(a,b;k,\alpha,\beta)$ forms as below;*

$$\begin{aligned}
G(a,b;k,\alpha,\beta)&=\left[\frac{1}{k}\sum_{i=1}^{k}\left(\frac{(k+1-i)a^{\alpha}+ib^{\alpha}}{k+1}\right)^{\frac{\beta}{\alpha}}\right]^{\frac{1}{\beta}},\\
G(a,b;k,0,\beta)&=\left[\frac{1}{k}\sum_{i=1}^{k}a^{\frac{(k+1-i)\beta}{k+1}}b^{\frac{i\beta}{k+1}}\right]^{\frac{1}{\beta}},\\
G(a,b;k,\alpha,0)&=\prod_{i=1}^{k}\left(\frac{(k+1-i)a^{\alpha}+ib^{\alpha}}{k+1}\right)^{\frac{1}{k\alpha}},\\
G(a,b;k,0,0)&=\sqrt{ab};
\end{aligned}\tag{1.6}$$

and

$$
\begin{aligned}
g(a,b;k,\alpha,\beta) &= \left[\frac{1}{k+1}\sum_{i=0}^{k}\left(\frac{(k-i)a^{\alpha}+ib^{\alpha}}{k}\right)^{\frac{\beta}{\alpha}}\right]^{\frac{1}{\beta}}, \\
g(a,b;k,0,\beta) &= \left[\frac{1}{k+1}\sum_{i=0}^{k} a^{\frac{(k-i)\beta}{k}} b^{\frac{i\beta}{k}}\right]^{\frac{1}{\beta}}, \\
g(a,b;k,\alpha,0) &= \prod_{i=0}^{k}\left(\frac{(k-i)a^{\alpha}+ib^{\alpha}}{k}\right)^{\frac{1}{(k+1)\alpha}}, \\
g(a,b;k,0,0) &= \sqrt{ab}.
\end{aligned} \tag{1.7}
$$

Definition 1.2. *[1] A convex function $f(\boldsymbol{x})$, where is said to be Schur convex if for $\boldsymbol{a} \geq \boldsymbol{b}$, then $f(\mathbf{a}) \geq f(\boldsymbol{b})$, where $\boldsymbol{a} = (a_1, ..., a_n)$, $\boldsymbol{b} = (b_1, ..., b_n)$ and $\boldsymbol{x} = (x_1, ..., x_n)$.*

2 Main results

In this section, the convexity and Schur convexity of Gnan mean for two variables are established.

Lemma 2.1. *Let $\boldsymbol{a} = (a_1, a_2), \boldsymbol{b} = (b_1, b_2)$ and if $\boldsymbol{a} \geq \boldsymbol{b}$, then $G(\boldsymbol{a}; k, \alpha, \beta) \geq G(\boldsymbol{b}; k, \alpha, \beta)$.*

Proof. For $\boldsymbol{a} \geq \boldsymbol{b}$, then $a_1 \geq b_1$ and $a_2 \geq b_2$.

Also, for $\alpha, \frac{\beta}{\alpha} > 0$, then the following inequality holds

$$\left[\tfrac{(k+1-i)a_1^{\alpha}+ia_2^{\alpha}}{k+1}\right]^{\frac{\beta}{\alpha}} \geq \left[\tfrac{(k+1-i)b_1^{\alpha}+ib_2^{\alpha}}{k+1}\right]^{\frac{\beta}{\alpha}},$$

implies that,

$$\left[\tfrac{1}{k}\textstyle\sum_{i=1}^{k}\left(\tfrac{(k+1-i)a_1^{\alpha}+ia_2^{\alpha}}{k+1}\right)^{\frac{\beta}{\alpha}}\right]^{\frac{1}{\beta}} \geq \left[\tfrac{1}{k}\textstyle\sum_{i=1}^{k}\left(\tfrac{(k+1-i)b_1^{\alpha}+ib_2^{\alpha}}{k+1}\right)^{\frac{\beta}{\alpha}}\right]^{\frac{1}{\beta}},$$

$$G(\boldsymbol{a}, k, \alpha, \beta) \geq G(\boldsymbol{b}, k, \alpha, \beta).$$

Thus the proof of Lemma 2.1 completes. □

Theorem 2.1. *The Gnan mean $G(a, b; k, \alpha, \beta)$ is convex if α, $\frac{\beta}{\alpha} \geq 1$ and concave if α, $\frac{\beta}{\alpha} \leq 1$.*

Proof. From Definition 1.1 , let

$$G^{\beta}(a,b;k,\alpha,\beta)=\left[\frac{1}{k}\sum_{i=1}^{k}\left(\frac{(k+1-i)a^{\alpha}+ib^{\alpha}}{k+1}\right)^{\frac{\beta}{\alpha}}\right] \tag{2.1}$$

with simple manipulation eqn (2.1) rewritten as,

$$kG^{\beta}(a,b;k,\alpha,\beta)=\sum_{i=1}^{k}\left(\frac{(k+1-i)a^{\alpha}+ib^{\alpha}}{k+1}\right)^{\frac{\beta}{\alpha}} \tag{2.2}$$

put $a=t$ and $b=1$, then eqn (2.2) takes the form

$$kG^{\beta}(t,1;k,\alpha,\beta)=\sum_{i=1}^{k}\left(\frac{(k+1-i)t^{\alpha}+i1^{\alpha}}{k+1}\right)^{\frac{\beta}{\alpha}} \tag{2.3}$$

differentiating both sides of the eqn (2.3) with respect to t twice, we have

$$\begin{aligned} k\frac{\partial^2}{\partial t^2}\left[G^{\beta}(t,1;k,\alpha,\beta)\right] &= \sum_{i=1}^{k}\beta(\beta-\alpha)\left(\frac{(k+1-i)t^{\alpha}+i1^{\alpha}}{k+1}\right)^{\frac{\beta}{\alpha}-2}\left(\frac{k+1-i}{k+1}t^{\alpha-1}\right)^{2} \\ &\quad+\sum_{i=1}^{k}\left(\frac{(k+1-i)t^{\alpha}+i1^{\alpha}}{k+1}\right)^{\frac{\beta}{\alpha}-1}\left(\frac{k+1-i}{k+1}t^{\alpha-2}\right) \end{aligned} \tag{2.4}$$

$$=\beta\sum_{i=1}^{k}\left[(\beta-\alpha)\frac{(k+1-i)t^{2\alpha-2}}{((k+1-i)t^{\alpha}+i)(k+1)}+(\alpha-1)\frac{k+1-i}{k+1})t^{\alpha-2}\right]\left(\frac{(k+1-i)a^{\alpha}+ib^{\alpha}}{k+1}\right)^{\frac{\beta}{\alpha}-1},$$

then $\quad \frac{\partial^2}{\partial t^2}\left[G^{\beta}(t,1;k,\alpha,\beta)\right]\geq 0,$

if $\quad (\alpha-1)\geq 0$ and $(\beta-\alpha)\geq 0.$

Hence the proof of Theorem 2.1. □

Theorem 2.2. *The Gnan mean $G(a,b;k,\alpha,\beta)$ is Schur convex if α, $\frac{\beta}{\alpha}\geq 1$.*

Proof. Consider the Gnan mean $G(a_1,a_2;k,\alpha,\beta)$.

Let $a_1=\lambda b_1+(1-\lambda)b_2$ and $a_2=(1-\lambda)b_1+\lambda b_2$, leads to

$$\begin{aligned} G(a_1,a_2;k,\alpha,\beta) &= G[\lambda b_1+(1-\lambda)b_2,(1-\lambda)b_1+\lambda b_2;k,\alpha,\beta] \\ &= G[\lambda(b_1,b_2)+(1-\lambda)(b_2,b_1);k,\alpha,\beta] \end{aligned}$$

since $G(a_1,a_2;k,\alpha,\beta)$ is convex, then

$$\leq \lambda G(b_1, b_2; k, \alpha, \beta) + (1 - \lambda) G(b_2, b_1; k, \alpha, \beta)$$

since $G(a_1, a_2; k, \alpha, \beta)$ is symmetric, then

$$= \lambda G(b_1, b_2; k, \alpha, \beta) + (1 - \lambda) G(b_1, b_2; k, \alpha, \beta)$$
$$= G(b_1, b_2; k, \alpha, \beta)$$

This proves the Schur convexity of Gnan mean $G(a, b; k, \alpha, \beta)$ for two variables. $\square$

Note: Similar arguments holds good for the Gnan mean in dual forms.

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