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Solution to an open problem by Rooin

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Abstract: In this note, we obtained the solution to an open problem posed by J. Rooin, using Levinson's Inequality.
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1 Introduction

In [2], for n arbitrary non-negative numbers $x_1, x_2, ..., x_n$, the un-weighted Arithmetic and Geometric means are defined respectively as follows;

$$A_n = \frac{1}{n}\sum_1^n x_i \qquad and \qquad G_n = \frac{1}{n}\prod_1^n (x_i)^{\frac{1}{n}} \tag{1.1}$$

and more over, for each $x_i \in [0, \frac{1}{2}]$, let $A_n^{/}$ and $G_n^{/}$ are the un-weighted Arithmetic and Geometric means of $1 - x_1, 1 - x_2, ..., 1 - x_n$ respectively defined as below;

$$A_n^{/} = \frac{1}{n}\sum_1^n (1 - x_i) \qquad and \qquad G_n^{/} = \frac{1}{n}\prod_1^n (1 - x_i)^{\frac{1}{n}}. \tag{1.2}$$

It is well known that the Arithmetic and Geometric means are the members of the family of Power mean. For un-weighted case the power mean in n variables is given by;

$$M_r = M_r(x_1, x_2, ..., x_n) = \left(\frac{1}{n}\sum_1^n (x_i)^r\right)^{\frac{1}{r}}; \qquad r \neq 0. \tag{1.3}$$

and

$$M_r^{/} = M_r^{/}(1-x_1, 1-x_2, ..., 1-x_n) = \left(\frac{1}{n}\sum_{1}^{n}(1-x_i)^r\right)^{\frac{1}{r}}; \qquad r \neq 0. \tag{1.4}$$

We recall the following definitions needed for this short note see [1].

Definition 1.1. *Let **a** $= (a_1, a_2, ..., a_n)$ and **w** $= (a_1, a_2, ..., a_n)$ are the two n-tuples, then the weighted Arithmetic mean of **a** and **w** is given by;*

$$A(\boldsymbol{a}, \boldsymbol{w}) = \frac{w_1 a_1 + ... + w_n a_n}{w_1 + ... + w_n} = \frac{1}{W_n}\sum_{i=1}^{n} w_i a_i, \tag{1.5}$$

where $W_n = w_1 + ... + w_n$

Definition 1.2. *Let **a** $= (a_1, a_2, ..., a_n)$ and **w** $= (a_1, a_2, ..., a_n)$ are the two n-tuples, then the weighted Geometric mean of **a** and **w** is given by;*

$$G(\boldsymbol{a}, \boldsymbol{w}) = \left(\prod_{i=1}^{n} a_i^{w_i}\right)^{\frac{1}{W_n}}, \tag{1.6}$$

where $W_n = w_1 + ... + w_n$

Definition 1.3. *[1] A function $f(x)$ is n-convex, $n \geq 2$ if and only if $f(x)^{n-2}$ exists and is convex.*

2 Solution to an open problem

In this section, we give an affirmative answer to query raised by J. Rooin in his paper [2], by using *Levinson's Inequality* [1].

Open Problem: For $x_i \in [0, \frac{1}{2}], i = 0, 1, ..., n$ and $k = 0, 1, 2, ...$, determine the values of the parameters r and s, so that the following inequality holds

$$\frac{(M_r)^x}{(-lnM_r)^k} - \frac{(M_s)^x}{(-lnM_s)^k} \geq \frac{(M_r)^x}{(-lnM_r)^k} - \frac{(M_s)^x}{(-lnM_s)^k}. \tag{2.1}$$

Lemma 2.1. *Let I be an interval in R and $M : I \longrightarrow R$ be 3- convex, **w** is a positive n-tuple, $n \geq 2$, **a** and **b** are n-tuples with elements in I and satisfying Max **a** $\leq$ Min **b** and*

$$a_1 + b_2 = ... = a_n + b_n \tag{2.2}$$

then

$$[A(M(\boldsymbol{a}), \boldsymbol{w}) - M(A(\boldsymbol{a}), \boldsymbol{w})] \leq [A(M(\boldsymbol{b}), \boldsymbol{w}) - M(A(\boldsymbol{b}), \boldsymbol{w})]. \tag{2.3}$$

If M is strictly 3-convex, then equality occurs in equation (2.3) if and only if **a** and **b** are constants conversely if for a continuous $M : I \longrightarrow R$ holds in equation (2.3), holds strictly for all positive 2-tuples **w** and all non-constants **a** and **b** with elements in I and satisfying the equation (2.2) with $n = 2$, then M is 3-convex, strictly 3-convex.

Lemma 2.2. *Let $n \geq 2$ and $\boldsymbol{a}$ and $\boldsymbol{b}$ are n-tuples with elements in I and satisfying the equation (2.2), $\boldsymbol{w}$ be another positive n-tuple. If $s > 0$ and $t < s$ or $t > 2s$ or $s = 0$ and $t > 0$ or $s < 0$ and $s > t > 2s$, then for $t \neq 0$,*

$$\left[(M_t(\boldsymbol{a},\boldsymbol{w}))^t - (M_s(\boldsymbol{a},\boldsymbol{w}))^t\right]^{\frac{1}{t}} \leq \left[(M_t(\boldsymbol{b},\boldsymbol{w}))^t - (M_s(\boldsymbol{b},\boldsymbol{w}))^t\right]^{\frac{1}{t}}, \tag{2.4}$$

if $s > 0$

$$\frac{G(\boldsymbol{a},\boldsymbol{w})}{G(\boldsymbol{b},\boldsymbol{w})} \leq \frac{M_s(\boldsymbol{a},\boldsymbol{w})}{M_s(\boldsymbol{b},\boldsymbol{w})}, \tag{2.5}$$

equality occurs in any two of these inequalities if and only if $\boldsymbol{a}$ or $\boldsymbol{b}$ are constants.

Now, we give the proof of the open problem by using above Lemmas.

Proof. With simple manipulations in equation (2.4), that is for un- weighted means by ignoring the power $\frac{1}{t}$ and also replace the power t by x, then the inequality (2.4) takes the form;

$$[(M_r(\boldsymbol{a}))^x - (M_s(\boldsymbol{a}))^x] \leq [(M_r(\boldsymbol{b}))^x - (M_s(\boldsymbol{b}))^x], \tag{2.6}$$

integrating both sides of inequality 2.6 with limits from x to ∞, we get

$$\frac{(M_r)^x}{(-lnM_r)} - \frac{(M_s)^x}{(-lnM_s)} \geq \frac{(M_r)^x}{(-lnM_r)} - \frac{(M_s)^x}{(-lnM_s)}. \tag{2.7}$$

Now by induction, that is by integrating both sides of inequality k times from x to ∞ and by setting $\boldsymbol{b} = (1 - \boldsymbol{a})$ or $\boldsymbol{a} + \boldsymbol{b} = 1$, then

$$\frac{(M_r)^x}{(-lnM_r)^k} - \frac{(M_s)^x}{(-lnM_s)^k} \geq \frac{(M_r)^x}{(-lnM_r)^k} - \frac{(M_s)^x}{(-lnM_s)^k}. \tag{2.8}$$

The inequality (2.8), holds for the parameters $s > 0$ and $t < s$ or $t > 2s$ or $s = 0$ and $t > 0$ or $s < 0$ and $s > t > 2s$. □

3 Some refinements

In this section, we established the following deduction and refinement.

Lemma 3.1. *Let $n \geq 2$, $\boldsymbol{a}$ is the n-tuple with $0 \leq \boldsymbol{a} \leq \frac{1}{2}$ and $r, s \in R, r < s$, then*

$$\frac{M_r(\boldsymbol{a})}{M_r(1-\boldsymbol{a})} \leq \frac{M_s(\boldsymbol{a})}{M_s(1-\boldsymbol{a})}, \tag{3.1}$$

if and only if $|r + s| < 3$ and $\frac{2^r}{r} \leq \frac{2^r}{r}$, if $r > 0$ and $r2^r > s2^s$ if $s < 0$.

In equation 3.1, put $r = \frac{1}{2}$ and $s = 1$, we have

$$\frac{M_{\frac{1}{2}}(\boldsymbol{a})}{M_{\frac{1}{2}}(1-\boldsymbol{a})} \leq \frac{A(\boldsymbol{a})}{A(1-\boldsymbol{a})}, \tag{3.2}$$

again if $s = \frac{1}{2}$ and $\boldsymbol{a} + \boldsymbol{b} = 1$, then equation (2.5) for un-weighted means

$$\frac{G(\boldsymbol{a})}{G(1-\boldsymbol{a})} \leq \frac{M_{\frac{1}{2}}(\boldsymbol{a})}{M_{\frac{1}{2}}(1-\boldsymbol{a})}, \tag{3.3}$$

combining inequalities (3.2) and (3.3), we get

$$\frac{G(\boldsymbol{a})}{G(1-\boldsymbol{a})} \leq \frac{M_{\frac{1}{2}}(\boldsymbol{a})}{M_{\frac{1}{2}}(1-\boldsymbol{a})} \leq \frac{A(\boldsymbol{a})}{A(1-\boldsymbol{a})}, \tag{3.4}$$

inequality (3.4), is the refinement and extension of the following well known inequality,

$$\frac{G(\boldsymbol{a})}{G(1-\boldsymbol{a})} \leq \frac{A(\boldsymbol{a})}{A(1-\boldsymbol{a})}. \tag{3.5}$$

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