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The structure of even powers in Z_3: Critical structural factors that prevent the formation of even-powered triples greater than squares

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Abstract: Integer structure analysis illustrates the critical structural factors which underpin the failure of ($N^{4m}+M^{4m}$) ever to equal an equivalent power. The number 3 plays a vital role as integers divisible by 3, when raised to an even power of the form 4*m*, have rows in a table of modular rings which are triangular numbers, whereas other integers raised to the same power have rows which are pentagonal numbers. The substructure within these sequences of pentagonal numbers is order within order, analogous to structure in chaos theory.

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1 Introduction

When the right-end digit (RED) structure of squares, N^2, is compared with that of N^{4m}, it is clear that the higher-powered structure is much more restricted (Tables 1, 2). Thus, a sum of squares produces seven possible REDs, but the sum ($N^{4m}+M^{4m}$) produces only three [2].

$(M^2)^* \downarrow \ (N^2)^* \rightarrow$	1	5	9
0	1	5	9
4	5	9	~~3~~
6	~~7~~	1	5

Table 1. Sums of squares of *N* (odd) and *M* (even)

REDs for the equation

$$N^{4m} + M^{4m} = Q^{4m} \tag{1.1}$$

will equal (1 + 0 = 1), (5 + 0 = 5) or (6 + 5 = 1). We shall use the modular ring Z_3 (Table 3) to illustrate how the integer structure prevents integer solutions for any of these RED systems. Hence Equation (1.1) is invalid.

$M^{4m}\downarrow \ \ N^{4m}\rightarrow$	1	5
0	1	5
6	~~7~~	1

Table 2. *m* = 1, 2, 3, 4

Function	$3r_0$	$3r_1+1$	$3r_2+2$
Row ↓ Class →	$\bar{0}_3$	$\bar{1}_3$	$\bar{2}_3$
0	0	1	2
1	3	4	5
2	6	7	8
3	9	10	11
4	12	13	14
5	15	16	17
6	18	19	20
7	21	22	23
8	24	25	26
9	27	28	29
10	30	31	32

Table 3. Z_3

2 RED Sets

(i) Structure (6 + 5 = 1)

We have recently shown [6] that the primitive Pythagorean triples (pPts) with this RED structure do not exist. Hence,

$$\left(N^{4m}\right)^2+\left(M^{4m}\right)^2\neq\left(Q^{4m}\right)^2 \tag{2.1}$$

for this RED structure.

(ii) Structure (0 + 5 = 5)

Similarly, since 5 will be a common factor, this RED structure fails for PPts and hence must fail for the in-equation (2.1).

(iii) Structure (1 + 0 = 1)

This RED structure is valid for pPts. The critical structural factors are that integers, N, with $3|N$ all fall in Class $\bar{0}_3$ (Table 3), and that there are no even powers in Class $\bar{2}_3$. Therefore, $N^{4m}, M^{4m}\notin\bar{1}_3$ because $\left(\bar{1}_3+\bar{1}_3\right)\in\bar{2}_3$, and so $\left(\bar{1}_3+\bar{0}_3\right)\in\bar{1}_3$ is the only possible structure for Equation (1.1). The system $\left(\bar{0}_3+\bar{0}_3\right)\in\bar{0}_3$ will have a common factor.

- When $(N^{4m})^* = 1$, if $\begin{cases}3\nmid N, & N^{4m}\in\bar{1}_3;\\ 3|N, & N^{4m}\in\bar{0}_3.\end{cases}$
- When $(M^{4m})^* = 0$, if $\begin{cases}3\nmid M. & M^{4m}\in\bar{1}_3;\\ 3|M, & M^{4m}\in\bar{0}_3.\end{cases}$

We then have two specific cases to consider, namely

- $N^{4m}\in\bar{0}_3, M^{4m}\in\bar{1}_3$,
- $N^{4m}\in\bar{1}_3, M^{4m}\in\bar{0}_3$.

(a) $N^{4m}\in\bar{0}_3, M^{4m}\in\bar{1}_3$:

When $\left(\left(N^{4m}\right)^*=1\wedge N^{4m}\in\bar{0}_3\right)\&\left(\left(M^{4m}\right)^*=0\wedge M^{4m}\in\bar{1}_3\right)$, $N^{4m}=3r_0\ \&\ M^{4m}=3r_1+1$.

Since $3|N\in\bar{0}_3$, even powered odd integers in this class have rows the elements of which belong to the sequence of triangular numbers, $\frac{1}{2}n(n+1)$ [2]. Hence,

$$r_0 = 3 + 12n(n+1) \tag{2.2}$$

and for t even [2]

$$r_1 = \tfrac{1}{3}\left(2t^2 - 1\right). \tag{2.3}$$

Since

$$3r_0 + 3r_1 + 1 = 3r_1' + 1, \tag{2.4}$$

$$r_0 + r_1 = r_1'. \tag{2.5}$$

The row r_1' of Q should follow the pentagonal numbers [2]. Hence,

$$r_1' = 8\left(\tfrac{1}{2}n'(3n'+1)\right) \tag{2.6}$$

For $(N^{4m})^*=1$ the other form $\frac{1}{2}n'(3n'-1)$ does not occur, and so Equation (2.5) becomes

$$t^2 + 4 + 18n(n+1) = 6n'(3n'+1). \tag{2.7}$$

Since $\frac{1}{3}\left(t^2+4\right)$ is non-integral, Equation (2.7) does not yield integer solutions.

(b) $N^{4m} \in \bar{1}_3, M^{4m} \in \bar{0}_3$:

When

$$\left(\left(N^{4m}\right)^* = 1 \wedge N^{4m} \in \bar{1}_3\right) \& \left(\left(M^{4m}\right)^* = 0 \wedge M^{4m} \in \bar{0}_3\right),$$

the structure is more complex as the rows of both N^{4m} and Q^{4m} are elements of the sequence of pentagonal numbers, Equation (2.5) becomes

$$r_1 + r_0 = r_1' \tag{2.8}$$

with

$$r_1 = 8\left(\tfrac{1}{2}n(3n+1)\right)$$

and

$$r_1' = 8\left(\tfrac{1}{2}n'(3n'+1)\right)$$

r_0 is even and $3|r_0$. Thus the equivalent of (2.7) becomes

$$r_0 = 4(n'-n)\left(3(n'+n)+1\right) \tag{2.9}$$

The row of M^{4m} has $3^s 2^s$ factors with a minimum s = 3 (m = 1). On the other hand, ($n' - n$) is deficient in factors of 3. To have factors of 3, n and n' must be in the same class (Table 4). This eliminates two thirds of the combinations.

n'	$\bar{1}_3$	$\bar{2}_3$	$\bar{0}_3$	$\bar{1}_3$	$\bar{1}_3$	$\bar{2}_3$	$\bar{2}_3$	$\bar{0}_3$	$\bar{0}_3$
n	$\bar{1}_3$	$\bar{2}_3$	$\bar{0}_3$	$\bar{0}_3$	$\bar{2}_3$	$\bar{0}_3$	$\bar{1}_3$	$\bar{1}_3$	$\bar{2}_3$
$n' - n$	$\bar{0}_3$	$\bar{0}_3$	$\bar{0}_3$	$\bar{1}_3$	$\bar{2}_3$	$\bar{2}_3$	$\bar{1}_3$	$\bar{2}_3$	$\bar{1}_3$

Table 4. Class structures

Some typical values of n for $N^4 \in \bar{1}_3$ are shown in Table 5. We can see multiples of the pentagonal numbers in the last column of Table 5, which appear in the Euler pentagonal number theorem [1]:

$$\prod_{n=1}^{\infty}\left(1-x^n\right) = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + \ldots,$$

which, in turn, can be related to recurrences [7].

N	$N^4 \in \bar{1}_3$	**Row = 8*K***	**K=** $\frac{1}{2}n(3n+1)$	n
7	2401	800	100	8
11	14641	4880	610	20
13	28561	9520	1190	28
17	83521	27840	3480	48
19	130321	43440	5430	60
23	279841	93280	11660	88
29	707281	235760	29470	140
31	923521	307840	38480	160
37	1874161	624720	78090	228
41	2825761	941920	117740	280
43	3418801	1139600	142450	308
47	4879681	1626560	203320	368
49	5764801	1921600	240200	400
53	7890481	2630160	328770	468
59	12117361	4039120	504890	580
61	13845841	4615280	576910	620

Table 5. Values of n for $N^4 \in \bar{1}_3$

The separation in Table 6 shows other functions, such as $m = 2 + 3t$ (Class $\bar{1}_3$). In this regard, the constraints that $(N^4)^*$ and $(Q^4)^* = 1$ should be kept in mind as t values might be inapplicable (Class $\bar{2}_3$ for example Table 6). In view of these restrictions it is understandable why factor 3 is limited for $(n' - n)$ so that integral solutions do not occur, which is expected from Fermat's Last Theorem. Powers higher than 4, N^{4m}, $m > 1$, have the same pattern.

Class *n*		$\frac{1}{4}n$	***m***	
			A	**B**
$\bar{1}_3$	28	7	2	
	88	22		4
	160	40	5	
	280	70		7
	400	100	8	
	580	145		10
$\bar{2}_3$	8	2	1	
	20	5		2
	140	35		5
	308	77	7	
	368	92		8
	620	155	10	
$\bar{0}_3$	48	12		3
	60	15	3	
	228	57	6	
	468	117		9

Table 6. $A = \frac{1}{2}m(3m+1)$; $B = \frac{1}{2}m(3m-1)$

3 Final Comments

Integer Structure Analysis (ISA) of many systems in number theory can often identify the constraints on solutions, particularly when seeking counter examples [3,4]. ISA can also reveal unexpected characteristics of systems. For example, the elements of pentagonal numbers can also belong to other sequences (Table 6) [7], analogous to the order within order of chaos theory. The results here apply to N^{4m}, for natural numbers, because of the RED structure.

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