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On a class of infinite sequences with relatively prime numbers and twin prime conjecture

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Abstract: In this paper, a class of infinite sequences with positive integer terms is considered. If the Twin Prime Conjecture, stipulating that there are infinitely many twin prime numbers, is true, then the sequence of all twin prime numbers belongs to the same class. In the present investigation, it is proved that the mentioned class is non-empty and moreover there exists at least one element of this class containing all twin primes.
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1 Introduction

Let $\mathbb{N}$ be the set of all positive integers; let $\gcd(u, v)$ be the greatest common divisor of u and v.

First, we start with the following two definitions.

Definition 1. *Let $a_1, a_2, a_3, \ldots,$ be an infinite sequence of strictly increasing positive integers. We call this sequence T-sequence (from twin sequence) if the following two conditions are fulfilled:*

1) For every $i, j \in \mathbb{N}$ such that $i \neq j$

$$\gcd(a_i, a_j) = 1 \tag{1}$$

2) $a_2 = a_1 + 2$ and for every $m \in \mathbb{N}, m > 1$, at least one of the equalities:

$$a_m = a_{m-1} + 2 \tag{2}$$

$$a_m = a_{m+1} - 2 \tag{3}$$

hold.

Definition 2. *The class of all T-sequences we denote by $\mathcal{T}$.*

Together with T-sequences, we will consider the class $\mathcal{M}$ of all finite sequences of strictly increasing positive integers.

Definition 3. *Let $n > 1$ and the sequence $a_1, a_2, \ldots, a_n$ belongs to $\mathcal{M}$. If this sequence satisfies conditions 1) and 2) for $m < n$ (from Definition 1) and also the relation $a_n = a_{n-1} + 2$, we call this sequence T^*- sequence.*

Definition 4. *The class of all T^*-sequences will be denoted by $\mathcal{T}^*$.*

As an obvious corollary from the above definitions we obtain:

Lemma 1. *All terms of a T-sequence or a T^*-sequence are odd numbers.*

2 Main results

The first main result of the paper is

Theorem 1. *$\mathcal{T}$ is a non-empty class.*

This theorem means that there exist at least one T-sequence.

Proof. Let p^* denote the twin prime sequence, i.e. the sequence: $3, 5, 7, 11, 13, 17, 19, 29, 31$, etc. (with general term further denoted by p^*_m). We have the alternative:

A) p^* is infinite;

B) p^* is finite.

Let A) hold. Then, the Twin Prime Conjecture (see [1]) is true and since the conditions 1) and 2) from Definition 1 are obviously satisfied for p^*, then p^* is a T-sequence, i.e. $p^* \in \mathcal{T}$ (see Definition 2). Therefore, $\mathcal{T}$ is a non-empty class. Thus, for case A) the theorem is proved.

Now, let case B) hold. Then, the Twin Prime Conjecture is false and there exists a maximal twin prime number p^*_n. Then, p^* is obviously a T^*-sequence. Let us suppose that Theorem 1 is not true. Let $b_1, b_2, \ldots, b_s$ is a T^*-sequence with length $s \geq n$ having the property $b_i = p^*_i$ for $i = 1, \ldots, n$. We denote by δ_s the product $\prod_{i=1}^{s} b_i$ and set:

$$b_{s+1} = 2 + \delta_s$$

$$b_{s+2} = 4 + \delta_s$$

From the above we have

$$b_{s+2} = b_{s+1} + 2$$

and

$$b_2 = p^*_2 = 5 = 3 + 2 = p^*_1 + 2 = b_1 + 2$$

Now, let us consider the sequence $b_1, b_2, \ldots, b_s, b_{s+1}, b_{s+2}$. We shall show that this sequence is a T^*-sequence, too. Indeed, if $i, j \leq s$ and $i \neq j$ then (1) is satisfied since $b_1, b_2, \ldots, b_s$ is a T^*-sequence. Let $i \leq s, j > s$ and $\gcd(b_i, b_j) = d$. Then, d is a divisor of δ_s. Hence, d is a divisor of 2 or 4. Let us suppose that $d > 1$. Then d is an even number and moreover b_i and b_j are also even. But the last contradicts to Lemma 1. Therefore, $d = 1$. Finally, it remains to consider the case $i = s+1, j = s+2$. Using again the denotation d for $\gcd(b_i, b_j)$ we obtain that d is a divisor of $b_j - b_i = 2$. Hence, $d = 1$ because of Lemma 1.

Thus, we proved that $b_1, b_2, \ldots, b_s, b_{s+1}, b_{s+2}$ is a T^*-sequence having the property $b_i = p^*_i, i = 1, 2, \ldots, n$. In the same way we are able to continue this new T^*-sequence (as far as we want). Thus, we obtain infinitely many T^*-sequences, each one containing the previous. Now, we take the union of all these sequences. It is easy to verify that this union is a T-sequence which contains the twin prime sequence.

Therefore, Theorem 1 is proved for the case B), too. □

We note that if the twin prime sequence is infinite, then it satisfies the conditions (2) and (3).

From the proof of Theorem 1 we obtain the second main result:

Theorem 2. *There exist at least one T-sequence that contains all twin primes.*

Here we propose the following

Open problem 1. *Try to give at least one example of a T-sequence in explicit form.*

Open problem 2. *Try to give an example of a T-sequence that does not contain prime numbers.*

3 Conclusion

An immediate generalization of the considered (see Definition 1) may be proposed through replacing the number 2 in (2) by $2^{\gamma(m)}$ and in (3) by $2^{\eta(m)}$ (and by imposing the restriction $a_2 = a_1 + 2^{\eta(1)}$), where $\gamma : \mathbb{N} \to \mathbb{N}$ and $\eta : \mathbb{N} \to \mathbb{N}$ are arbitrary mappings preserving the strict monotonic increase of the infinite sequence $a_1, a_2, \ldots,$ and the Property 1) from Definition 1, too. This type of sequences we call generalized T-sequences, or in short GT-sequences. By analogy, one may also introduce generalized T^*-sequences (or in short GT^*-sequences). In this case, it must be fulfilled:

$$a_n = a_{n-1} + 2^{\gamma(n)} \text{ (see Definition 3)}$$

In the same way that Theorem 1 was proved, one may prove the following

Theorem 3. *The class of all GT-sequences is non-empty.*

This theorem is an obvious generalization of Theorem 1.

Finally, we must note that just as T-sequences are related to the twin prime sequence, the GT-sequences are related to appropriate sequences of primes that we call generalized twin prime sequences. Each term of such a sequence differs by a suitable positive power of number 2 from at least one of its adjacent terms. Thus, we come to the question for generalization of the famous Twin Prime Conjecture, this time for the above mentioned sequences of primes. Also we see that the following theorem must be true.

Theorem 4. *For every generalized twin prime sequence there exists at least one GT-sequence which contains it.*

References

[1] Guy, R. *Unsolved Problems in Number Theory*, 3rd ed. New York, Springer-Verlag, 2004, pp. 32.