NNTDM 17 (2011), 3, 13-14

Note on φ, ψ and σ-functions. Part 3

Krassimir T. Atanassov

Department of Bioinformatics and Mathematical Modelling
Institute of Biophysics and Biomedical Engineering – Bulgarian Academy of Sciences,
Acad. G. Bonchev Str., Bl. 105, Sofia-1113, Bulgaria
e-mail: $krat@bas.bg$

Abstract: Equalities connecting φ, ψ and σ-functions are formulated and proved.
Keywords: Arithmetic functions φ, ψ and σ.
AMS Classification: 11A25

The paper is a continuation of [1, 2]. For the well-known functions φ, ψ and σ (see, e.g. [3]), here we prove the following
Theorem. For each natural number $n > 1$

$$n^{2n} < \varphi(n)^{\varphi(n)}\psi(n)^{\psi(n)} \leq \varphi(n)^{\varphi(n)}\sigma(n)^{\sigma(n)}. \quad (1)$$

Proof. From the obvious inequality $\psi(n) \leq \sigma(n)$ for each natural number $n \geq 2$, it follows directly, that

$$\varphi(n)^{\varphi(n)}\psi(n)^{\psi(n)} \leq \varphi(n)^{\varphi(n)}\sigma(n)^{\sigma(n)}. \quad (2)$$

Therefore, we must prove only the first inequality of (1).
Let $n = p$ be a prime number. Then, we check directly, that

$$\begin{aligned} &(p-1)^{p-1}(p+1)^{(p+1)} - p^{2p} \\ &= (p^2-1)^{p-1}(p+1)^2 - p^{2p} \\ &> (p^{2(p-1)} - (p-1)p^{2(p-1)-2})(p^2+2p+1) - p^{2p} \\ &= p^{2p} - (p-1)p^{2p-2} + 2p^{2p-1} - 2(p-1)p^{2p-3} + p^{2p-2} - (p-1)p^{2p-4} - p^{2p} \\ &= -p^{2p-1} + p^{2p-2} + 2p^{2p-1} - 2p^{2p-2} + 2p^{2p-3} + p^{2p-2} - p^{2p-3} + p^{2p-4} \\ &= p^{2p-1} + p^{2p-3} + p^{2p-4} > 0. \end{aligned}$$

For the proof below, we need the following
Lemma. For each natural number $n \geq 2$

$$\varphi(n) + \psi(n) \geq 2n.$$

Proof. Let n be a prime number. Then, the assertion is obvious. Let the assertion be valid for some natural number n and let p be a prime number. If $p \notin \underline{set}(n)$, then

$$\varphi(np)+\psi(np)-2np = \varphi(p)(p-1)+\psi(n)(p+1)-2np = p(\varphi(n)+\psi(n)-2n)+\psi(n)-\varphi(n) \geq 0$$

by induction assumption. If $p \in \underline{set}(n)$, then

$$\varphi(np) + \psi(np) - 2np = \varphi(p)p + \psi(n)p - 2np = p(\varphi(n) + \psi(n) - 2n) \geq 0$$

by induction assumption. Now, we return to the proof of the Theorem.

Let us assume that

$$n^{2n} \leq \varphi(n)^{\varphi(n)} \psi(n)^{\psi(n)} \quad (3)$$

for some natural number $n \geq 2$. Let p be a prime number. For it, there two cases.

Case 1: $p \notin \underline{set}(n)$. Then,

$$X \equiv \varphi(np)^{\varphi(np)} \psi(np)^{\psi(np)} - (np)^{2np}$$

$$= (\varphi(n)(p-1))^{\varphi(n)(p-1)} (\psi(n)(p+1))^{\psi(n)(p+1)} - n^{2np} p^{2np}$$

$$= \varphi(n)^{\varphi(n)(p-1)} \psi(n)^{\psi(n)(p+1)} (p-1)^{\varphi(n)(p-1)} (p+1)^{\psi(n)(p+1)} - n^{2np} p^{2np}$$

$$= (\varphi(n)^{\varphi(n)} \psi(n)^{\psi(n)})^{p-1} \psi(n)^{2\psi(n)(p+1)} ((p-1)^{(p-1)} (p+1)^{(p+1)})^{\varphi(n)} . (p+1)^{(p+1)(\psi(n)-\varphi(n))} - n^{2np} p^{2np}$$

$$> (\varphi(n)^{\varphi(n)} \psi(n)^{\psi(n)})^{p} ((p-1)^{(p-1)} (p+1)^{(p+1)})^{\varphi(n)} . (p+1)^{(p+1)(\psi(n)-\varphi(n))} - n^{2np} p^{2np}$$

(by induction assumption)

$$> n^{2np} (((p-1)^{(p-1)} (p+1)^{(p+1)})^{\varphi(n)} . (p+1)^{(p+1)(\psi(n)-\varphi(n))} - p^{2np})$$

$$= n^{2np} (((p-1)^{(p-1)} (p+1)^{(p+1)})^{\varphi(n)} . (p+1)^{(p+1)(n-\varphi(n))} . (p+1)^{(p+1)(\psi(n)-n)} - p^{2np})$$

$$> n^{2np} ((p-1)^{(p-1)\varphi(n)} (p+1)^{(p+1)\varphi(n)} . (p-1)^{(p-1)(n-\varphi(n))} . (p+1)^{(p+1)(\psi(n)-n)} - p^{2np})$$

(by the Lemma)

$$> n^{2np} ((p-1)^{p-1} (p+1)^{(p+1)})^{n} - (p^{2p})^{n}) > 0.$$

Case 2: $p \in \underline{set}(n)$. Then

$$X = \varphi(np)^{\varphi(np)} \psi(np)^{\psi(np)} - (np)^{2np}$$

$$= (\varphi(n)p)^{\varphi(n)p} (\psi(n)p)^{\psi(n)p} - (np)^{2np}$$

$$= (\varphi(n))^{\varphi(n)p} p^{\varphi(n)p} (\psi(n))^{\psi(n)p} p^{\psi(n)p} - n^{2np} p^{2np}$$

$$= ((\varphi(n))^{\varphi(n)} (\psi(n))^{\psi(n)})^{p} p^{(\varphi(n)+\psi(n))p} - n^{2np} p^{2np} > 0$$

by the induction assumption and Lemma.

So, (3) is proved. From the validity of (2) and (3), the validity of (1) follows directly.

References

[1] Atanassov, K. Note on φ, ψ and σ functions. *Notes on Number Theory and Discrete Mathematics*, Vol. 12, 2006, No. 4, 23–24.

[2] Atanassov, K. Note on φ, ψ and σ-functions. Part 2. *Notes on Number Theory and Discrete Mathematics*, Vol. 16, 2010, No. 3, 25–28.

[3] Nagell, T. Introduction to number theory. *John Wiley & Sons*, New York, 1950.