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The relation between $\pi(x)$ and certain arithmetic functions

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Abstract: We prove an improvement Rosser-Schoenfeld inequalities, more precisely:

$$\frac{x^{k+1}}{(k+1)\log x + 0,1k^2 + 0,1k - 0,99} < A_k(x) < \frac{x^{k+1}}{(k+1)\log x - 0,1k^2 - 0,2k - 1,11},$$

where $A_k(x) = \sum_{p\leq x} p^k$ and $k \geq 0$.
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1 Introduction

In [5, Th2, page 69] appear the inequalities, which we will call inequalities of Rosser-Schoenfeld type. These inequalities have been improved in [3, p. 3], namely

$$\pi(x) > \frac{x}{\log x - 1}, \quad \text{if} \quad x \geq 5393, \tag{1}$$

$$\pi(x) < \frac{x}{\log x - 1.1}, \quad \text{if} \quad x \geq 60184, \tag{2}$$

We extend the Rosser-Schoenfeld inequality for the function, given by Jean-Pierre Massias and Guy Robin (see [4]). In our proof, the $\pi(x)$ function will appear. Throughout this paper, p denotes a prime number. For each $k \geq 0$, we define the functions $A_k : (0,\infty) \to (0,\infty)$ by

$$A_k(x) = \sum_{p\leq x} p^k \tag{3}$$

Observe that $A_0(x) = \sum\limits_{p\leq x} 1 = \pi(x)$.

We evaluate the asymptotic functions $A_k(x)$. For this we turn to some known results and for the proof of the next propositions (see [2], chap. V, 10, pages 228-232). We recall also a well known notation, see [2].

Let $g:(a,\infty)\to R$ be a function such that $g(x)\neq 0, \forall x\in(a,\infty)$. If $f:(b,\infty)\to R$ is a function, we write $f(x)\sim g(x)$ if $x\to\infty$ or simply $f(x)\sim g(x)$ if and only if $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)}=1$.

Proposition 1. Let $a>0, g:[a,\infty)\to(0,\infty)$ be a twice differentiable function with the property that there exists:

$$\lim_{x\to\infty}\frac{xg^{'}(x)}{g(x)}=\lambda\in\bar{\mathbb{R}}-\{-1\}$$

if $\lambda\in(-1,\infty)$ then $\int_a^\infty g(x)dx$ is divergent and

$$\int_a^\infty g(x)dx\sim\frac{1}{\lambda+1}xg(x),\quad if\quad x\to\infty.$$

If:

$$\lim_{x\to\infty}\frac{xg^{'}(x)}{g(x)}=-1,$$

we define $h(x)=xg(x)$ and $\varphi(y)=h(e^y)$. Then

$$\int_a^x g(t)dt=\int_{\log a}^{\log x}\varphi(y)dy.$$

Proposition 2. Let $a>0$, $f:[a,\infty)\to\mathbb{R}$, $g:[a,\infty)\to(0,\infty)$ be two Riemann integrable functions for any interval $[a,u]$, $\forall a\leq u<\infty$.
Suppose

$$f(x)\backsim g(x)\ \text{for}\ x\to\infty$$

and $\int_a^\infty g(t)\,dt$ is divergent, then

$$\int_a^x f(t)\,dt\backsim\int_a^x g(t)\,dt\ \text{for}\ x\to\infty.$$

To prove the next result (see [5], page 67):

Proposition 3. Let $f:(0,\infty)\to\mathbb{R}$ be a function which has a continuous derivative. Then

$$\sum_{p\leq x}f(p)=\pi(x)f(x)-\int_2^x\pi(t)f^{'}(t)dt\ \text{for}\ x\geq 2.$$

Taking in this proposition $f:(0,\infty)\to\mathbb{R}\ f(x)=x^k$ we get

Corollary 1. For each $k\geq 0$, we have the following equality

$$A_k(x)=\pi(x)x^k-k\int_2^x\pi(t)t^{k-1}dt\ \text{ for }\ x\geq 2.$$

Next we want a formula for asymptotic evaluation functions $A_k(x)$

2 Asymptotic evaluations

Lemma 1. For $k\geq 0$, we have

$$\int_2^x\frac{t^k}{\log t}dt\sim\frac{1}{k+1}\cdot\frac{x^{k+1}}{\log x}$$

Proof. We use Proposition 1, in which we take $g(x) = \frac{x^k}{\log x}$. Then $g^{'}(x) = \frac{kx^{k-1}\log x - x^{k-1}}{\log^2 x}$ and

$$\lim_{x\to\infty} \frac{xg^{'}(x)}{g(x)} = \lim_{x\to\infty} \frac{x^k(k\log x - 1)}{\log^2 x} \cdot \frac{\log x}{x^k} = k.$$

We obtain $\int_2^x \frac{t^k}{\log t} dt \sim \frac{1}{k+1} \cdot \frac{x^{k+1}}{\log x}$.

Theorem 1. The following asymptotic evaluation holds: If $k \geq 0$ then

$$A_k(x) \sim \frac{x^{k+1}}{(k+1)\log x}$$

Proof. We use the Prime Number Theorem

$$\pi(x) \sim \frac{x}{\log x} \Leftrightarrow \lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1.$$

From Corollary 1 we have

$$A_k(x) = \pi(x)x^k - k\int_2^x \pi(t)t^{k-1}dt \ \text{ for } x \geq 2.$$

We multiply this relation with $\frac{\log x}{x^{k+1}}$ and obtain

$$\frac{A_k(x)\log x}{x^{k+1}} = \frac{\pi(x)\log x}{x} - \frac{k\log x}{x^{k+1}}\int_2^x \pi(t)t^{k-1}dt.$$

We wish to prove that the second term limit from the right side is $\frac{1}{k+1}$.
From Prime Number Theorem we have

$$\pi(t)t^{k-1} \sim \frac{t}{\log t} \cdot t^{k-1} = \frac{t^k}{\log t}$$

From Proposition 2 we have

$$\int_2^x \pi(t)t^{k-1}dt \sim \int_2^x \frac{t^k}{\log t}dt$$

Given Lemma 1,

$$\lim_{x\to\infty} \frac{\int_2^x \frac{t^k}{\log x}dt}{\frac{x^{k+1}}{\log x}} = \lim_{x\to\infty} \frac{\frac{1}{k+1}\cdot\frac{x^{k+1}}{\log x}}{\frac{x^{k+1}}{\log x}} = \frac{1}{k+1},$$

we obtain that

$$\lim_{x\to\infty} \frac{A_k(x)\log x}{x^{k+1}} = \frac{1}{k+1} \text{ if and only if } A_k(x) \sim \frac{x^{k+1}}{(k+1)\log x} \blacksquare$$

3 Bordering for the functions $A_k(x)$

We want to generalize to $A_k(x)$ Rosser-Schoenfeld inequalities, more precisely:

$$\frac{x^{k+1}}{(k+1)(\log x - a)} < A_k(x) < \frac{x^{k+1}}{(k+1)(\log x - b)}$$

where $k \geq 0$ and $x \geq 60184$.

From Corollary 1 we have

$$A_k(x) = \pi(x)x^k - k\int_2^x \pi(t)t^{k-1}dt$$

We use relations (1) and (2), hence

$$A_k(x) < \frac{x^{k+1}}{\log x - 1,1} - k\int_2^{5393} \pi(t)t^{k-1}dt - k\int_{5393}^x \frac{t^k dt}{\log t - 1}$$

$$A_k(x) > \frac{x^{k+1}}{\log x - 1} - k\int_2^{60184} \pi(t)t^{k-1}dt - k\int_{60184}^x \frac{t^k dt}{\log t - 1,1}.$$

For we need to prove the desired inequality

$$\frac{x^{k+1}}{\log x - 1,1} - k\int_2^{5393} \pi(t)t^{k-1}dt - k\int_{5393}^x \frac{t^k dt}{\log t - 1} < \frac{x^{k+1}}{(k+1)(\log x - a)}$$

$$\frac{x^{k+1}}{\log x - 1} - k\int_2^{60184} \pi(t)t^{k-1}dt - k\int_{60184}^x \frac{t^k dt}{\log t - 1,1} > \frac{x^{k+1}}{(k+1)(\log x - b)}$$

The next sentence will give us the condition for the constants a and b such that the derivative of the function to be negative, respectively positive. We expect these constants to be dependent of k, and also, the function depending on k and inequalities will be satisfied from a certain rank depending on k.

Proposition 4. For $k \geq 0$ we consider $f_a : [5393, \infty) \to R$,

$$f_a(x) = \frac{x^{k+1}}{\log x - 1,1} - k\int_{5393}^x \frac{t^k dt}{\log t - 1} - \frac{x^{k+1}}{(k+1)(\log x - a)}$$

and, $g_b : [60184, \infty) \to R$,

$$g_b(x) = \frac{x^{k+1}}{\log x - 1} - k\int_{60184}^x \frac{t^k dt}{\log t - 1,1} - \frac{x^{k+1}}{(k+1)(\log x - b)}$$

Then, $f_a^{'}(x) < 0$ for $a > \frac{0,1k^2+0,2k+1,1}{k+1}$ and $g_b^{'}(x) > 0$ for $b < \frac{-0,1k^2-0,1k+1}{k+1}$, $\forall x \geq x_k$ and $k \geq 0$.

Proof.

$$f_a^{'}(x) = \frac{x^k(k\log x + \log x - 1,1k - 2,1)}{(\log x - 1,1)^2} - \frac{kx^k}{\log x - 1} - \frac{x^k(k\log x + \log x - ak - a - 1)}{(k+1)(\log x - a)^2}$$

We want: $f_a^{'}(x) < 0$, more accurate:

$$\frac{\log^2 x + 0,1k\log x - 3,1\log x + 2,1 - 0,11k}{\log^3 x - 3,2\log^2 x + 3,41\log x - 1,21}$$

$$-\frac{k\log x + \log x - ak - a - 1}{k\log^2 x + \log^2 x - 2ak\log x - 2a\log x + ka^2 + a^2} < 0$$

$$\Leftrightarrow (\log^2 x + 0,1k\log x - 3,1\log x + 2,1 - 0,11k)(k\log^2 x + \log^2 x - 2ak\log x - 2a\log x + ka^2 + a^2)$$

$$-(\log^3 x - 3,2\log^2 x + 3,41\log x - 1,21)(k\log x + \log x - ak - a - 1) < 0$$

$$\Leftrightarrow \log^3 x(0,1k^2 - ak + 0,2k - a + 1,1)$$
$$+\log^2 x(-0,11k^2 + a^2k - 0,2ak^2 + a^2 + 2,8ak + 3a - 1,42k - 4,51)$$
$$+\log x(0,1a^2k^2 - 3a^2k + 0,22ak^2 - 0,57ak - 3,1a^2 + 1,21k - 0,79a + 4,62)$$
$$-0,11k^2a^2 + 1,99a^2k - 1,21ak + 2,1a^2 - 1,21a - 1,21 < 0$$

We put the condition

$$0,1k^2 - ak + 0,2k - a + 1,1 < 0 \Leftrightarrow a > \frac{0,1k^2 + 0,2k + 1,1}{k+1}$$

For example: $a = \frac{0,1k^2+0,2k+1,11}{k+1}$. We obtain

$$-0,01\log^3 x - \frac{0,01k^4 - 0,17k^3 + 0,63k^2 + 1,778k - 0,0521}{k+1}\log^2 x$$
$$+\frac{0,001k^5 - 0,005k^4 - 0,1108k^3 + 0,4934k^2 + 3,78611k - 0,07641}{k+1}\log x$$
$$-\frac{-0,0011k^5 + 0,0166k^4 - 0,06582k^3 + 0,13836k^2 - 1,998231k + 0,03431}{k+1} < 0$$

for $x \geq x_k$.

Logarithmic function and power function is increasing function starting with a certain rank above relationship is satisfied.

$$g_b'(x) = \frac{x^k(k\log x + \log x - k - 2)}{\log^2 x - 2\log x + 1} - \frac{kx^k}{\log x - 1,1} - \frac{x^k(k\log x + \log x - kb - b - 1)}{(k+1)(\log^2 x - 2b\log x + b^2)}.$$

We want $g_b'(x) > 0$, more precisely

$$\frac{\log^2 x - 0,1k\log x - 3,1\log x + 0,1k + 2,2}{\log^3 x - 3,1\log^2 x + 3,2\log x - 1,1}$$
$$-\frac{k\log x + \log x - kb - b - 1}{k\log^2 x + \log^2 x - 2bk\log x - 2b\log x + kb^2 + b^2} > 0$$
$$\Leftrightarrow (\log^2 x - 0,1k\log x - 3,1\log x + 0,1k + 2,2)(k\log^2 x + \log^2 x - 2bk\log x - 2b\log x + kb^2 + b^2)$$
$$-(k\log x + \log x - kb - b - 1)(\log^3 x - 3,1\log^2 x + 3,2\log x - 1,1) > 0$$
$$\Leftrightarrow \log^3 x(-0,1k^2 - bk - b - 0,1k + 1) + \log^2 x(kb^2 + b^2 + 0,2bk^2 + 3,3kb + 3,1b + 0,1k^2 - 0,9k - 4,1)$$
$$+\log x(-0,1k^2b^2 - 3,2kb^2 - 0,2bk^2 - 3,1b^2 - 1,4kb - 1,2b + 1,1k + 4,3)$$
$$+0,1k^2b^2 + 2,3kb^2 - 1,1kb + 2,2b^2 - 1,1b - 1,1 > 0.$$

We put the condition

$$-0,1k^2 - bk - b - 0,1k + 1 > 0 \Leftrightarrow b < \frac{-0,1k^2 - 0,1k + 1}{k+1}.$$

For example: $b = \frac{-0,1k^2-0,1k+0,99}{k+1}$. We obtain:

$$0,01\log^3 x + \frac{(-0,01k^4 - 0,23k^3 - 1,43k^2 - 2,241k - 0,0509)\log^2 x}{k+1}$$

$$+\frac{(-0,001k^5-0,013k^4+0,1168k^3+1,7646k^2+4,64979k+0,07369)\log x}{k+1}$$
$$+\frac{0,001k^5+0,024k^4+0,1352k^3-0,2134k^2-2,41659k-0,03278}{k+1}>0$$

for $x \geq x_k$.

Similarly to the previous relationship to a certain rank, depending on k, the relationship is real.

Proposition 5. For $a \geq 2$ and $\forall c \in R$ we have

$$\lim_{x\to\infty}\frac{\int_a^x \frac{t^k dt}{(\log x-c)^3}}{\frac{x^{k+1}}{\log^2 x}}=0.$$

Proof. From Proposition 1 we have $g(x)=\frac{x^k}{(\log x-c)^3}$

$$g^{'}(x)=\frac{x^{k-1}(k\log x-kc-3)}{(\log x-c)^4}$$

$$\lim_{x\to\infty}\frac{x^k(k\log x-kc-3)}{(\log x-c)^4}\cdot\frac{(\log x-c)^3}{x^k}=k.$$

Then,

$$\int_a^x\frac{t^k dt}{(\log t-c)^3}\sim\frac{1}{k+1}\cdot\frac{x^{k+1}}{(\log x-c)^3}.$$

We have

$$\lim_{x\to\infty}\frac{\int_a^x \frac{t^k dt}{(\log x-c)^3}}{\frac{x^{k+1}}{\log^2 x}}=\lim_{x\to\infty}\frac{\log^2 x}{(k+1)(\log x-c)^3}=0.\blacksquare$$

Proposition 6. For $k \geq 0$, we consider $f:[5393,\infty)\to R$,

$$f(x)=\frac{x^{k+1}}{\log x-1,1}-k\int_{5393}^x\frac{t^k dt}{\log t-1}-\frac{x^{k+1}}{(k+1)\log x-0,1k^2-0,2k-1,11}$$

and $g:[60184,\infty)\to R$,

$$g(x)=\frac{x^{k+1}}{\log x-1}-k\int_{60184}^x\frac{t^k dt}{\log t-1,1}-\frac{x^{k+1}}{(k+1)\log x+0,1k^2+0,1k-0,99}.$$

Then $\lim_{x\to\infty} f(x)=-\infty$ and $\lim_{x\to\infty} g(x)=\infty$.

Proof.

$$\int_{5393}^x\frac{t^k dt}{\log t-1}=\frac{t^{k+1}}{(k+1)(\log t-1)}|_{5393}^x+\frac{1}{k+1}\int_{5393}^x\frac{t^k dt}{(\log t-1)^2}$$
$$=\frac{x^{k+1}}{(k+1)(\log x-1)}-\frac{5393^{k+1}}{(k+1)(\log 5393-1)}$$
$$+\frac{t^{k+1}}{(k+1)^2(\log t-1)^2}|_{5393}^x+\frac{2}{(k+1)^2}\int_{5393}^x\frac{t^k dt}{(\log t-1)^3}$$
$$=\frac{x^{k+1}}{(k+1)(\log x-1)}-\frac{5393^{k+1}}{(k+1)(\log 5393-1)}$$

$$+\frac{x^{k+1}}{(k+1)^2(\log x-1)^2}-\frac{5393^{k+1}}{(k+1)^2(\log 5393-1)^2}$$

$$+\frac{2}{(k+1)^2}\int_{5393}^{x}\frac{t^k dt}{(\log t-1)^3}.$$

We note that

$$A=\frac{5393^{k+1}}{(k+1)(\log 5393-1)}+\frac{5393^{k+1}}{(k+1)^2(\log 5393-1)^2}.$$

Then

$$\lim_{x\to\infty}[\frac{x^{k+1}}{\log x-1,1}-\frac{kx^{k+1}}{(k+1)(\log x-1)}+kA$$

$$-\frac{kx^{k+1}}{(k+1)^2(\log x-1)^2}-\frac{x^{k+1}}{(k+1)\log x-0,1k^2-0,2k-1,11}$$

$$-\frac{2k}{(k+1)^2}\int_{5393}^{x}\frac{t^k dt}{(\log t-1)^3}]$$

$$=\lim_{x\to\infty}\{[\frac{(-0,01k-0,01)\log^2 x}{(k\log^2 x+\log^2 x-0,1k^2\log x-1,3k\log x-2,21\log x+0,11k^2+0,22k+1,221)}$$

$$+\frac{(-0,01k^4+0,17k^3-0,631k^2+0,219k+0,02)\log x}{(k\log^2 x+\log^2 x-0,1k^2\log x-1,3k\log x-2,21\log x+0,11k^2+0,22k+1,221)}$$

$$+\frac{0,01k^4-0,18k^3+0,711k^2-0,22k-0,01}{(k\log^2 x+\log^2 x-0,1k^2\log x-1,3k\log x-2,21\log x+0,11k^2+0,22k+1,221)}].$$

$$\frac{x^{k+1}}{(k^2\log^2 x+2k\log^2 x+\log^2 x-2k^2\log x-4k\log x-2\log x+k^2+2k+1)}+kA$$

$$-\frac{2k}{(k+1)^2}\int_{5393}^{x}\frac{t^k dt}{(\log^3 t-1)^3}\}.$$

Considering Proposition 5, we have

$$\lim_{x\to\infty}\frac{x^{k+1}}{\log^2 x}\cdot[\frac{-0,01k-0,01}{(k+1)(k^2+2k+1)}+\frac{kA\log^2 x}{x^{k+1}}$$

$$-\frac{2k}{(k+1)^2}\cdot\frac{\int_{5393}^{x}\frac{t^k dt}{(\log^3 t-1)^3}}{\frac{x^{k+1}}{\log^2 x}}]=-\infty$$

$$\int_{60184}^{x}\frac{t^k dt}{\log t-1,1}=\frac{t^{k+1}}{(k+1)(\log t-1,1)}|_{60184}^{x}+\frac{1}{k+1}\int_{60184}^{x}\frac{t^k dt}{(\log t-1,1)^2}$$

$$=\frac{x^{k+1}}{(k+1)(\log x-1,1)}-\frac{60184^{k+1}}{(k+1)(\log 60184-1,1)}$$

$$+\frac{t^{k+1}}{(k+1)^2(\log t-1,1)^2}|_{60184}^{x}+\frac{2}{(k+1)^2}\int_{60184}^{x}\frac{t^k dt}{(\log t-1,1)^3}$$

$$=\frac{x^{k+1}}{(k+1)(\log x-1,1)}-\frac{60184^{k+1}}{(k+1)(\log 60184-1,1)}+\frac{x^{k+1}}{(k+1)^2(\log x-1,1)^2}$$

$$-\frac{60184^{k+1}}{(k+1)^2(\log 60184-1,1)^2}+\frac{2}{(k+1)^2}\int_{60184}^{x}\frac{t^k dt}{(\log t-1,1)^3}.$$

We note that

$$B=\frac{60184^{k+1}}{(k+1)(\log 60184-1,1)}+\frac{60184^{k+1}}{(k+1)^2(\log 60184-1,1)^2}$$

$$\lim_{x\to\infty} g(x)=\lim_{x\to\infty}[\frac{x^{k+1}}{\log x-1}-\frac{x^{k+1}}{(k+1)\log x+0,1k^2+0,1k-0,99}$$

$$-\frac{kx^{k+1}}{(k+1)(\log x-1,1)}-\frac{kx^{k+1}}{(k+1)^2(\log x-1,1)^2}-\frac{2k}{(k+1)^2}\int_{60184}^{x}\frac{t^k dt}{(\log t-1,1)^3}+Bk]$$

$$\lim_{x\to\infty}\{[\frac{(0,01k+0,01)\log^2 x}{k\log^2 x+\log^2 x+0,1k^2\log x-0,9k\log x-1,99\log x-0,1k^2-0,1k+0,99}$$

$$+\frac{(-0,11k^4-1,319k^3-3,609k^2-1,332k-0,022)\log x}{k\log^2 x+\log^2 x+0,1k^2\log x-0,9k\log x-1,99\log x-0,1k^2-0,1k+0,99}$$

$$+\frac{-0,0979k^4-0,0837k^3+1,11651k^2+0,11352k+0,00121}{k\log^2 x+\log^2 x+0,1k^2\log x-0,9k\log x-1,99\log x-0,1k^2-0,1k+0,99}]$$

$$\cdot\frac{x^{k+1}}{k^2\log^2 x+2k\log^2 x+\log^2 x-2,2k^2\log x-4,4k\log x-2,2\log x+0,121k^2+0,242k+0,121}$$

$$+kB-\frac{2}{(k+1)^2}\int_{60184}^{x}\frac{t^k dt}{(\log t-1,1)^3}]\}.$$

Considering Proposition 5, we have

$$\lim_{x\to\infty}\frac{x^{k+1}}{\log^2 x}[\frac{0,01k+0,01}{(k+1)(k^2+2k+1)}+\frac{kB\log^2 x}{x^{k+1}}$$

$$-\frac{\frac{2}{(k+1)^2}\int_{60184}^{x}\frac{t^k dt}{(\log t-1,1)^3}}{\frac{x^{k+1}}{\log^2 x}}]=\infty.\blacksquare$$

Using Propositions 4 and 6 we obtain:

Theorem 2. For x big enough and $k\geq 0$ we have the following relation

$$\frac{x^{k+1}}{(k+1)\log x+0,1k^2+0,1k-0,99}<A_k(x)<\frac{x^{k+1}}{(k+1)\log x-0,1k^2-0,2k-1,11}.$$

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