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Remark on Jacobsthal numbers. Part 2

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The n-th Jacobsthal number ($n \geq 0$) is defined by

$$J_n = \frac{2^n - (-1)^n}{3}$$

(see, e.g., [1]).

The first ten members of the sequence $\{J_n\}$ are

0	1	2	3	4	5	6	7	8	9
0	1	1	3	5	11	21	43	85	171

Now, we generalize these numbers to the form:

$$J_n^s = \frac{s^n - (-1)^n}{s+1},$$

where $n \geq 0$ is a natural number and $s \geq 0$ is a real number.

Obviously, when $s = 2$ we obtain the standard Jacobsthal numbers.

The first five members of the sequence $\{J_n^s\}$ with respect to n are

0	1	2	3	4
0	1	$s-1$	$s^2 - s + 1$	$s^3 - s^2 + s - 1$

In the case $s = 0$ we obtain

$$J_n^0 = -(-1)^n = (-1)^{n+1}.$$

In the case $s = 1$ we obtain

$$J_n^1 = \frac{1 - (-1)^n}{2}$$

and the first ten members of sequence $\{J_n^1\}$ with respect to n are

0	1	2	3	4	5	6	7	8	9
0	1	0	1	0	1	0	1	0	1

In the case $s = 3$ we obtain

$$J_n^1 = \frac{3^n - (-1)^n}{4}$$

and the first ten members of sequence $\{J_n^3\}$ with respect to n are

0	1	2	3	4	5	6	7	8	9
0	1	2	7	20	61	182	547	1640	4921

Theorem 1. For every natural number $n \geq 0$ and real number $s \geq 0$

$$J_{n+1}^s = s.J_n^s + (-1)^n.$$

Proof. Directly it can be checked that for each $n \geq 0$:

$$\begin{aligned} J_{n+1}^s &= \tfrac{s^{n+1}-(-1)^{n+1}}{s+1} \\ &= \tfrac{s.s^n+(-1)^n}{s+1} \\ &= \tfrac{s.(s^n-(-1)^n)+s.(-1)^n+(-1)^n}{s+1} \\ &= s.J_n^s + (-1)^n. \end{aligned}$$

The next step of generalization of the Jacobsthal numbers has the form:

$$J_n^{s,t} = \frac{s^n - (-t)^n}{s+t},$$

where $n \geq 0$ is a natural number and $s \neq -t$ are arbitrary real numbers.

It is possible to consider also the case $s = -t$. In this case we define

$$J_n^{-t,t} = \lim_{s \to -t} \frac{s^n - (-t)^n}{s+t}.$$

For the right side of this equality we apply the L'Hopital's rule and obtain

$$\begin{aligned} J_n^{-t,t} &= \tfrac{\frac{d}{ds}(s^n-(-t)^n)|_{s=-t}}{\frac{d}{ds}(s+t)|_{s=-t}} \\ &= \tfrac{n.s^{n-1}|_{s=-t}}{1|_{s=-t}} \\ &= n.(-t)^{n-1}. \end{aligned}$$

Respectilevy,

$$J_n^{s,-s} = n.s^{n-1}.$$

We can prove, as above

Theorem 2. For every natural number $n \geq 0$ and real numbers s, t

$$J_{n+1}^{s,t} = s.J_n^{s,t} + (-t)^n.$$

Proof. Let $s \neq -t$. Then it can be directly checked that for each $n \geq 0$:

$$\begin{aligned} J_{n+1}^{s,t} &= \tfrac{s^{n+1}-(-t)^{n+1}}{s+t} \\ &= \tfrac{s.s^n+t.(-t)^n}{s+t} \\ &= \tfrac{s.(s^n-(-t)^n)+s.(-t)^n+t.(-t)^n}{s+t} \\ &= s.J_n^{s,t} + (-t)^n. \end{aligned}$$

When $s = -t$, then

$$J_{n+1}^{s,-s} = (n+1).s^n = n.s^{n-1}.s + s^n = s.J_n^{s,-s} + (-(-s))^n.$$

The theorem is proved.

Finally, we mention the following equalities.

$$J_n^{0,0} = 0,$$

$$J_n^{1,-1} = n,$$

$$J_n^{s,0} = s^{n-1},$$

$$J_n^{0,t} = (-t)^{n-1}$$

$$J_n^{s,-1} = s^{n-1} + s^{n-2} + ... + 1.$$

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References

[1] Ribenboim, P. *The Theory of Classical Variations,* Springer, New York, 1999.