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On a limit involving the product of prime numbers

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Abstract. Let p_k denote the kth prime number. The aim of this note is to prove that the limit of the sequence $(p_n/\sqrt[n]{p_1 \cdots p_n})$ is e.

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1 Introduction

Let p_n denote the nth prime number. The famous prime number theorem asserts that

$$p_n \sim n \log n \text{ as } n \to \infty, \tag{1.1}$$

i.e. $\lim\limits_{n\to\infty} \dfrac{p_n}{n \log n} = 1$. There are many consequences of (1.1). As immediate applications, we can deduce

$$\frac{p_{n+1}}{p_n} \to 1, \tag{1.2}$$

$$\frac{\log p_n}{\log n} \to 1, \tag{1.3}$$

as $n \to \infty$. From (1.2) or (1.3) easily follows

$$\sqrt[n]{p_n} \to 1. \tag{1.4}$$

Various limits, including e.g.

$$\frac{n^{\log p_{n+1}}}{(n+1)\log p_n} \to 1, \tag{1.5}$$

are induced in [4] (see pp. 247–254), where the unsolved conjecture of the first author, i.e.

$$\frac{p_{n+1} - p_n}{\sqrt{p_n}} \to 0, \tag{1.6}$$

is also stated.

The aim of this paper is to study the limit of $p_n/\sqrt[n]{p_1\cdots p_n}$, and to show in fact that

$$\frac{p_n}{\sqrt[n]{p_1\cdots p_n}} \to e. \tag{1.7}$$

One of the main ingredients will be the use of a certain inequality involving Chebyshev's function

$$\theta(x) = \sum_{p\le x} \log p,$$

where p runs through the primes $\le x$.

2 Main results

We need also the following limit relation:

Lemma 2.1.

$$\log p_n - \frac{p_n}{n} \to 1 \text{ as } n \to \infty. \tag{2.1}$$

Proof. By a result of P. Dusart [1] one has

$$p_n = n(\log n + \log\log n - 1) + n\cdot\theta(n), \tag{2.2}$$

where $\theta(n) > 0$ and $\theta(n) \to 0$ as $n \to \infty$. Thus

$$\begin{aligned}\log p_n - \frac{p_n}{n} &= \log\big(nf(n) + n\theta(n)\big) - f(n) - \theta(n) = \\ &= \log\left(1 + \frac{\log\log n - 1 + \theta(n)}{\log n}\right) + 1 - \theta(n) = \\ &= \log\left(1 + \frac{\log\log n - 1 + \theta(n)}{\log n}\right) + 1 - \theta(n) \to 1,\end{aligned}$$

since $\dfrac{\log\log n - 1 + \theta(n)}{\log n} \to 0$. Here $f(n) = \log n + \log\log n - 1$. For details see also the first author's paper [3]. □

The following result is due to Rosser and Schoenfeld (see [2]).

Lemma 2.2. *There exists a positive constant $c > 0$ such that for all $x > 1$ one has*

$$|\theta(x) - x| < c\cdot\frac{x}{\log^2 x}. \tag{2.3}$$

The main result of this paper is contained in the following:

Theorem 2.1. *The relation* (1.7) *holds true.*

Proof. Put $A_n = p_n / \sqrt[n]{p_1 \cdots p_n}$. Then, by definition of function $\theta(x)$, one can write

$$\log A_n = \log p_n - \frac{1}{n}\theta(p_n) = \log p_n - \frac{p_n}{n} + \frac{1}{n}\big(p_n - \theta(p_n)\big).$$

By (2.1) it will be sufficient to show that

$$\frac{p_n - \theta(p_n)}{n} \to 0. \tag{2.4}$$

Now, by (2.3) of Lemma 2.2 one gets $\frac{|\theta(p_n) - p_n|}{n} < \frac{c \cdot p_n}{n \log^2 p_n}$. By (1.1) and (1.3) one has $\frac{p_n}{n \log^2 p_n} \sim \frac{1}{\log p_n}$, so clearly $\frac{|\theta(p_n) - p_n|}{n} \to 0$. This implies (2.4), and the proof of Theorem 2.1 is finished, as $\log A_n \to 1$ implies $A_n \to e$, (i.e.) relation (1.7) holds true. □

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