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A novel approach to the discovery of ternary BBP-type formulas for polylogarithm constants

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Abstract: Using clear and straightforward approaches, we prove new ternary (base 3) digit extraction BBP-type formulas for polylogarithm constants. Some known results are also rediscovered in a more direct and elegant manner. An hitherto unproved degree 4 ternary formula is also proved. Finally, a couple of ternary zero relations are established, which prove two known but hitherto unproved formulas.
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1 Introduction

BBP-type formulas are formulas of the form

$$\alpha = \sum_{k=0}^{\infty} 1/b^k \sum_{j=1}^{n} a_j/(kn+j)^s$$

where s, b, n and a_j are integers, and α is some constant. Formulas of this type were first introduced in a 1996 paper [1], where a formula of this type for π was given. Such formulas have the remarkable property that they permit one to calculate base-b digits of the constant α beginning at an arbitrary starting position, by means of a simple algorithm that requires almost no memory and (depending on how many digits are required) without the need for multiple-precision arithmetic software [2]. Such formulas also have intriguing connections to the age-old problem of understanding why the digits of various transcendental constants appear "normal" – each string of m-long digits appears, in the limit, with frequency $1/b^m$ [3, 4, 5, 2].

While many binary BBP-type formulas are now known, only relatively few ternary (base-3) BBP-type formulas have been discovered. This present paper is concerned with the symbolic (that is, non-computer-search-based) discovery of ternary (base-3) BBP-type formulas for polylogarithm constants. The methods used here aim to complement the experimental approaches that have dominated the area. Through fundamental methods, a wide range of interesting formulas will be obtained. In most cases, the procedure for obtaining the ternary formulas shall consist mainly of evaluating a polylogarithm functional equation at indicated coordinates and noting the following identities for the real and imaginary parts of the polylogarithm function:

$$\begin{aligned}\text{Re Li}_s\left[pe^{ix}\right] &= \sum_{k=1}^{\infty}\frac{p^k\cos kx}{k^s}\\ \text{Im Li}_s\left[pe^{ix}\right] &= \sum_{k=1}^{\infty}\frac{p^k\sin kx}{k^s}\,,\end{aligned}\tag{1}$$

for $p \in [0,1]$, $x \in \mathbb{R}$ and $s \in \mathbb{Z}^+$. In the above equations, Li is the notation for the polylogarithm function defined by

$$\text{Li}_s[z] = \sum_{k=1}^{\infty}\frac{z^k}{k^s},\quad |z| \le 1\,.$$

When $p = 1$ we have

$$\begin{aligned}\text{Li}_{2n}[e^{ix}] &= \text{Gl}_{2n}(x) + i\text{Cl}_{2n}(x)\\ \text{Li}_{2n+1}[e^{ix}] &= \text{Cl}_{2n+1}(x) + i\text{Gl}_{2n+1}(x)\,,\end{aligned}\tag{2}$$

where Gl and Cl are Clausen sums [6] defined, for $n \in \mathbb{Z}^+$ by

$$\begin{aligned}\text{Cl}_{2n}(x) &= \sum_{k=1}^{\infty}\frac{\sin kx}{k^{2n}}, \quad \text{Cl}_{2n+1}(x) = \sum_{k=1}^{\infty}\frac{\cos kx}{k^{2n+1}}\\ \text{Gl}_{2n}(x) &= \sum_{k=1}^{\infty}\frac{\cos kx}{k^{2n}}, \quad \text{Gl}_{2n+1}(x) = \sum_{k=1}^{\infty}\frac{\sin kx}{k^{2n+1}}\,.\end{aligned}\tag{3}$$

We shall find the following formulas useful:

$$\begin{aligned}\text{Gl}_{2n}(x) &= (-1)^{1+[n/2]}2^{n-1}\pi^n\text{B}_n(x/2\pi)/n!\\ \frac{1}{m^{n-1}}\text{Cl}_n(mx) &= \sum_{r=0}^{m-1}\text{Cl}_n(x + 2\pi r/m)\,.\end{aligned}\tag{4}$$

Here $[n/2]$ denotes the integer part of $n/2$ and B_n are the Bernoulli polynomials defined by

$$\frac{te^{xt}}{\text{e}^t - 1} = \sum_{n=0}^{\infty}\frac{\text{B}_n(x)t^n}{n!}\,.$$

2 Degree 1 Ternary BBP-type Formulas

In reference [7], several degree 1 BBP-type formulas in general bases are proven. In many of the formulas, ternary formulas may be readily obtained by writing the base in each case as a power of 3.

Here we now present a couple of interesting degree 1 ternary formulas.

The following identities are easily verified:

$$\mathrm{Li}_1\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{6}\right)\right] = \frac{1}{2}\ln 3 + \frac{i\pi}{6}$$

$$\text{and} \tag{5}$$

$$\mathrm{Li}_1\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{2}\right)\right] = \frac{1}{2}\ln 3 - \ln 2 + \frac{i\pi}{6}\,.$$

We therefore have the formulas:

$$\ln 2 = \mathrm{Re}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{6}\right)\right] - \mathrm{Re}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{2}\right)\right]\,, \tag{6}$$

$$\ln 3 = 2\mathrm{Re}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{6}\right)\right] \tag{7}$$

and

$$\pi = 6\,\mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{6}\right)\right] = 6\,\mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{2}\right)\right]\,. \tag{8}$$

It is also straightforward to verify that:

$$\ln 2 = \mathrm{Li}_1\left[\frac{1}{3}\right] - \mathrm{Li}_1\left[-\frac{1}{3}\right]$$

$$\text{and} \tag{9}$$

$$\ln 3 = 2\mathrm{Li}_1\left[\frac{1}{3}\right] - \mathrm{Li}_1\left[-\frac{1}{3}\right]\,.$$

Based on the above formulas, we are now ready to derive explicit BBP-type formulas for $\ln 2$, $\ln 3$ and π.

2.1 Ternary formulas for $\ln 2$

Using the first of Eq. (1), we note that

$$\begin{aligned}\mathrm{Re}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{6}\right)\right] &= \sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{3}}\right)^k\frac{\cos(k\pi/6)}{k}\\ &= \frac{1}{2\cdot 3^6}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}\left[\frac{3^6}{12k+1}+\frac{3^5}{12k+2}\right.\\ &\quad -\frac{3^4}{12k+4}-\frac{3^4}{12k+5}-\frac{2\cdot 3^3}{12k+6}-\frac{3^3}{12k+7}\\ &\quad \left.-\frac{3^2}{12k+8}+\frac{3}{12k+10}+\frac{3}{12k+11}+\frac{2}{12k+12}\right]\end{aligned} \tag{10}$$

and

$$\text{Re Li}_1\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{2}\right)\right] = \frac{1}{3^6}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}\left[-\frac{3^5}{12k+2}+\frac{3^4}{12k+4}\right.$$
$$\left.-\frac{3^3}{12k+6}+\frac{3^2}{12k+8}-\frac{3}{12k+10}+\frac{2}{12k+12}\right]. \tag{11}$$

Subtracting Eq. (11) from Eq. (10) in accordance with Eq. (6), we obtain the following ternary BBP-type formula for $\ln 2$:

$$\ln 2 = \frac{1}{2\cdot 3^5}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}\left[\frac{3^5}{12k+1}+\frac{3^5}{12k+2}\right.$$
$$-\frac{3^4}{12k+4}-\frac{3^3}{12k+5}-\frac{3^2}{12k+7}-\frac{3^2}{12k+8} \tag{12}$$
$$\left.+\frac{3}{12k+10}+\frac{1}{12k+11}\right].$$

Note that an alternating version of Eq. (12), using the same scheme, is

$$\ln 2 = \frac{1}{18}\sum_{k=0}^{\infty}\left(-\frac{1}{27}\right)^k\left[\frac{9}{6k+1}+\frac{9}{6k+2}-\frac{3}{6k+4}-\frac{1}{6k+5}\right].$$

From the first of Eq. (9), we can obtain yet another ternary formula for $\ln 2$, as follows:

$$\text{Li}_1\left[\frac{1}{3}\right] = \frac{1}{9}\sum_{k=0}^{\infty}\frac{1}{9^k}\left[\frac{3}{2k+1}+\frac{1}{2k+2}\right] \tag{13}$$

and

$$\text{Li}_1\left[-\frac{1}{3}\right] = \frac{1}{9}\sum_{k=0}^{\infty}\frac{1}{9^k}\left[-\frac{3}{2k+1}+\frac{1}{2k+2}\right] \tag{14}$$

Subtracting Eq. (14) from Eq. (13) in accordance with the first of Eq. (9), we obtain

$$\ln 2 = \frac{3}{2}\sum_{k=0}^{\infty}\frac{1}{9^k}\left[\frac{1}{2k+1}\right], \tag{15}$$

which is listed in the BBP Compendium as formula (48).

2.2 Ternary formulas for $\ln 3$

Using Eq. (10) and Eq. (7), we obtain the following ternary BBP-type formula for $\ln 3$:

$$\ln 3 = \frac{1}{3^6}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}\left[\frac{3^6}{12k+1}+\frac{3^5}{12k+2}\right.$$
$$-\frac{3^4}{12k+4}-\frac{3^4}{12k+5}-\frac{2\cdot 3^3}{12k+6}-\frac{3^3}{12k+7} \tag{16}$$
$$\left.-\frac{3^2}{12k+8}+\frac{3}{12k+10}+\frac{3}{12k+11}+\frac{2}{12k+12}\right].$$

An alternating version of the above formula is

$$\ln 3 = \frac{1}{27}\sum_{k=0}^{\infty}\left(-\frac{1}{27}\right)^k\left[\frac{27}{6k+1}+\frac{9}{6k+2}-\frac{3}{6k+4}-\frac{3}{6k+5}-\frac{2}{6k+6}\right].$$

Combining Eq. (13) and Eq. (14) according to the second equality of Eq. (9) we obtain another ternary BBP-type formula for $\ln 3$ as :

$$\ln 3 = \frac{1}{9}\sum_{k=0}^{\infty}\frac{1}{9^k}\left[\frac{9}{2k+1}+\frac{1}{2k+2}\right], \tag{17}$$

which is Formula (51) of the Compendium.

2.3 Ternary formulas for π

From Eq. (8), we immediately obtain the ternary BBP-type formulas

$$\begin{aligned}\pi = \frac{\sqrt{3}}{3^5}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}\bigg[&\frac{3^5}{12k+1}+\frac{3^5}{12k+2}+\frac{2\cdot 3^4}{12k+3}\\ &+\frac{3^4}{12k+4}+\frac{3^3}{12k+5}-\frac{3^2}{12k+7}-\frac{3^2}{12k+8}\\ &-\frac{2\cdot 3}{12k+9}-\frac{3}{12k+10}-\frac{1}{12k+11}\bigg]\end{aligned} \tag{18}$$

and

$$\begin{aligned}\pi = \frac{2\sqrt{3}}{3^5}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}\bigg[&\frac{3^5}{12k+1}-\frac{3^4}{12k+3}+\frac{3^3}{12k+5}\\ &-\frac{3^2}{12k+7}+\frac{3}{12k+9}-\frac{1}{12k+11}\bigg].\end{aligned} \tag{19}$$

An alternating version of Eq. (19) is

$$\pi = 2\sqrt{3}\sum_{k=0}^{\infty}\left(-\frac{1}{3}\right)^k\left[\frac{1}{2k+1}\right]. \tag{20}$$

2.4 Ternary Zero Relations

Eq. (15) may be rewritten in base 3^6, length 12 as

$$\ln 2 = \frac{4}{3^5}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}\left[\frac{3^4}{12k+2}+\frac{3^2}{12k+6}+\frac{1}{12k+10}\right] \tag{21}$$

Subtracting Eq. (21) from Eq. (12), we obtain the following ternary zero relation:

$$0 = \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \left[\frac{3^5}{12k+1} - \frac{5 \cdot 3^4}{12k+2} - \frac{3^4}{12k+4} \right.$$
$$- \frac{3^3}{12k+5} - \frac{2^3 \cdot 3^2}{12k+6} - \frac{3^2}{12k+7} - \frac{3^2}{12k+8} \tag{22}$$
$$\left. - \frac{5}{12k+10} + \frac{1}{12k+11} \right] .$$

Note that

$$\text{Eq. (22)} = \text{Compendium formula (104)} - \text{Compendium formula (103)} = 0 . \tag{23}$$

Subtracting Eq. (18) from Eq. (19), we obtain the zero relation:

$$0 = \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \left[\frac{243}{12k+1} - \frac{243}{12k+2} - \frac{324}{12k+3} \right.$$
$$- \frac{81}{12k+4} + \frac{27}{12k+5} - \frac{9}{12k+7} + \frac{9}{12k+8} \tag{24}$$
$$\left. + \frac{12}{12k+9} + \frac{3}{12k+10} - \frac{1}{12k+11} \right] .$$

Note also that

$$\text{Eq. (24)} = \text{Compendium formula (104)} + \text{Compendium formula (103)} = 0 \tag{25}$$

Eqs. (23) and (25) therefore establish that

$$\begin{aligned} \text{Compendium formula(103)} &= 0 \\ \text{and} & \\ \text{Compendium formula(104)} &= 0 . \end{aligned} \tag{26}$$

Thus the hitherto unproved formulas (103) and (104) in the BBP Compendium are now proved.

3 Degree 2 Ternary BBP-type Formulas

The dilogarithm reflection formula (Eq. A.2.1.7 of [6]) is

$$\frac{\pi^2}{6} - \ln x \ln(1-x) = \mathrm{Li}_2[x] + \mathrm{Li}_2[1-x] .$$

Putting $x = -\exp(i\pi/3)$ in the above equation and taking real and imaginary parts gives

$$\frac{5\pi^2}{72} - \frac{1}{8} \ln^2 3 = \mathrm{Re}\,\mathrm{Li}_2 \left[\frac{1}{\sqrt{3}} \exp\left(\frac{i\pi}{6} \right) \right] \tag{27}$$

and

$$\frac{2}{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{\pi\ln 3}{12} = \mathrm{Im}\,\mathrm{Li}_2\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{6}\right)\right] . \tag{28}$$

A two-variable functional equation for dilogarithms, due to Kummer (Eq. A.2.1.19 of [6]) is

$$\begin{aligned}\mathrm{Li}_2\left[\frac{x(1-y)^2}{y(1-x)^2}\right] &= \mathrm{Li}_2\left[-\frac{x(1-y)}{(1-x)}\right] + \mathrm{Li}_2\left[-\frac{(1-y)}{y(1-x)}\right] \\ &+ \mathrm{Li}_2\left[\frac{x\,(1-y)}{y\,(1-x)}\right] + \mathrm{Li}_2\left[\frac{1-y}{1-x}\right] + \frac{1}{2}\ln^2 y\,.\end{aligned} \tag{29}$$

Choosing $x = -\exp(i\pi/3)$ and $y = \exp(i\pi/3)$ in Eq. (29) gives

$$\frac{\pi^2}{12} - \frac{\ln^2 3}{4} = \mathrm{Li}_2\left[\frac{1}{3}\right] - \frac{1}{2}\mathrm{Li}_2\left[-\frac{1}{3}\right] . \tag{30}$$

Note that the choice of $x = -1$ and $y = 1/3$ gives the same result.

Another two-variable functional equation for dilogarithms, due to Abel (Eq. A.2.1.16 of [6]) is

$$\begin{aligned}\mathrm{Li}_2\left[\frac{x}{1-x}\cdot\frac{y}{1-y}\right] &= \mathrm{Li}_2\left[\frac{x}{(1-y)}\right] + \mathrm{Li}_2\left[\frac{y}{(1-x)}\right] \\ &- \mathrm{Li}_2\left[x\right] - \mathrm{Li}_2\left[y\right] - \ln(1-x)\ln(1-y)\,.\end{aligned} \tag{31}$$

Choosing $x = -\exp(i\pi/3)$ and $y = \exp(-i\pi/3)$ in Eq. (31) and taking imaginary parts gives

$$\frac{5}{2}\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{\pi\ln 3}{4} = 3\mathrm{Im}\,\mathrm{Li}_2\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{2}\right)\right] . \tag{32}$$

3.1 Ternary Formula for π^2

Solving Eq. (27) and Eq. (30) for π^2, we obtain

$$\pi^2 = 36\mathrm{Re}\,\mathrm{Li}_2\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{6}\right)\right] - 18\mathrm{Li}_2\left[\frac{1}{3}\right] + 9\mathrm{Li}_2\left[-\frac{1}{3}\right] . \tag{33}$$

Writing

$$\begin{aligned}\operatorname{Re}\operatorname{Li}_2\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{6}\right)\right] = \frac{1}{2\cdot 3^6}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}&\left[\frac{3^6}{(12k+1)^2}+\frac{3^5}{(12k+2)^2}\right.\\&-\frac{3^4}{(12k+4)^2}-\frac{3^4}{(12k+5)^2}-\frac{3^3}{(12k+6)^2}\\&-\frac{3^3}{(12k+7)^2}-\frac{3^2}{(12k+8)^2}\\&\left.+\frac{3}{(12k+10)^2}+\frac{2}{(12k+12)^2}\right]\end{aligned} \tag{34}$$

and

$$\begin{aligned}\operatorname{Li}_2\left[\pm\frac{1}{3}\right] = \frac{2^2}{3^6}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}&\left[\pm\frac{3^5}{(12k+2)^2}+\frac{3^4}{(12k+4)^2}\right.\\&\pm\frac{3^3}{(12k+6)^2}+\frac{3^2}{(12k+8)^2}\\&\left.\pm\frac{3}{(12k+10)^2}+\frac{1}{(12k+12)^2}\right],\end{aligned} \tag{35}$$

and combining them according to Eq. (33) gives the ternary BBP-type formula for π^2 as

$$\begin{aligned}\pi^2 = \frac{2}{27}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}&\left[\frac{3^5}{(12k+1)^2}-\frac{5\cdot 3^4}{(12k+2)^2}\right.\\&-\frac{3^4}{(12k+4)^2}-\frac{3^3}{(12k+5)^2}-\frac{2^3\cdot 3^2}{(12k+6)^2}\\&-\frac{3^2}{(12k+7)^2}-\frac{3^2}{(12k+8)^2}\\&\left.-\frac{5}{(12k+10)^2}+\frac{1}{(12k+11)^2}\right].\end{aligned} \tag{36}$$

Incidentally, Eq. (36) is Formula (57) of the Compendium.

3.2 Ternary Formula for $\ln^2 3$

Solving Eq. (27) and Eq. (30) for $\ln^2 3$, we obtain

$$\ln^2 3 = 12\operatorname{Re}\operatorname{Li}_2\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{6}\right)\right] - 10\operatorname{Li}_2\left[\frac{1}{3}\right] + 5\operatorname{Li}_2\left[-\frac{1}{3}\right]. \tag{37}$$

Using Eqs. (34) and (35) above in Eq. (37), we obtain the ternary BBP-type formula for $\ln^2 3$ as

$$
\begin{aligned}
\ln^2 3 = \frac{1}{3^6}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}\Bigg[&\frac{2\cdot 3^7}{(12k+1)^2}-\frac{2\cdot 3^8}{(12k+2)^2}\\
&-\frac{2\cdot 13\cdot 3^4}{(12k+4)^2}-\frac{2\cdot 3^5}{(12k+5)^2}-\frac{2^3\cdot 3^5}{(12k+6)^2}\\
&-\frac{2\cdot 3^4}{(12k+7)^2}-\frac{2\cdot 13\cdot 3^2}{(12k+8)^2}\\
&-\frac{2\cdot 3^4}{(12k+10)^2}+\frac{2\cdot 3^2}{(12k+11)^2}-\frac{8}{(12k+12)^2}\Bigg],
\end{aligned} \tag{38}
$$

which is Formula (58) of the Compendium.

3.3 Ternary Formula for $\pi \ln 3$

Solving Eqs. (28) and (32) for $\pi \ln 3$ gives

$$
\pi \ln 3 = 48 \operatorname{Im} \operatorname{Li}_2\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{2}\right)\right] - 60 \operatorname{Im} \operatorname{Li}_2\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{6}\right)\right]. \tag{39}
$$

Now

$$
\begin{aligned}
\operatorname{Im} \operatorname{Li}_2\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{6}\right)\right] = \frac{\sqrt{3}}{54}\sum_{k=0}^{\infty}\left(-\frac{1}{27}\right)^k\Bigg[&\frac{9}{(6k+1)^2}+\frac{9}{(6k+2)^2}\\
&+\frac{6}{(12k+3)^2}+\frac{3}{(12k+4)^2}+\frac{1}{(6k+5)^2}\Bigg]
\end{aligned} \tag{40}
$$

and

$$
\operatorname{Im} \operatorname{Li}_2\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{2}\right)\right] = \frac{\sqrt{3}}{27}\sum_{k=0}^{\infty}\left(-\frac{1}{27}\right)^k\left[\frac{9}{(6k+1)^2}-\frac{3}{(6k+3)^2}+\frac{1}{(6k+5)^2}\right] \tag{41}
$$

Using Eqs. (40) and (41) in Eq. (39) leads to the ternary BBP-type formula

$$
\begin{aligned}
\pi \ln 3 = \frac{2\sqrt{3}}{3}\sum_{k=0}^{\infty}\left(-\frac{1}{27}\right)^k\Bigg[&\frac{9}{(6k+1)^2}-\frac{15}{(6k+2)^2}-\frac{18}{(6k+3)^2}\\
&-\frac{5}{(6k+4)^2}+\frac{1}{(6k+5)^2}\Bigg].
\end{aligned} \tag{42}
$$

3.4 Ternary Formula for $\operatorname{Cl}_2(\pi/3)$

Solving Eqs. (28) and (32) for $\operatorname{Cl}_2(\pi/3)$ gives

$$
\operatorname{Cl}_2\left(\frac{\pi}{3}\right) = 6 \operatorname{Im} \operatorname{Li}_2\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{2}\right)\right] - 6 \operatorname{Im} \operatorname{Li}_2\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{6}\right)\right]. \tag{43}
$$

Using Eqs. (40) and (41) in Eq. (43) leads to the ternary BBP-type formula

$$
\begin{aligned}
\mathrm{Cl}_2\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{9}\sum_{k=0}^{\infty}\left(-\frac{1}{27}\right)^k \Bigg[&\frac{9}{(6k+1)^2} - \frac{9}{(6k+2)^2} - \frac{12}{(6k+3)^2} \\
&-\frac{3}{(6k+4)^2} + \frac{1}{(6k+5)^2}\Bigg]\,.
\end{aligned} \tag{44}
$$

4 Degree 3 Ternary BBP-type Formulas

A functional identity for trilogarithms (Eq. A.2.6.10 of [6]) is

$$
\begin{aligned}
\mathrm{Li}_3\left[\frac{1-x}{1+x}\right] - \mathrm{Li}_3\left[\frac{x-1}{x+1}\right] &= 2\,\mathrm{Li}_3\left[1-x\right] + 2\,\mathrm{Li}_3\left[\frac{1}{1+x}\right] \\
&\quad - \frac{1}{2}\,\mathrm{Li}_3\left[1-x^2\right] - \frac{7}{4}\,\zeta(3) \\
&\quad + \frac{\pi^2}{6}\ln(1+x) - \frac{1}{3}\ln^3(1+x)\,.
\end{aligned} \tag{45}
$$

The use of $x=2$ in the above equation gives

$$
\frac{13}{6}\zeta(3) - \frac{1}{6}\pi^2\ln 3 + \frac{1}{6}\ln^3 3 = 2\,\mathrm{Li}_3\left[\frac{1}{3}\right] - \mathrm{Li}_3\left[-\frac{1}{3}\right]\,. \tag{46}
$$

Putting $x = \exp i\pi/3$ in the functional equation and taking real and imaginary parts gives

$$
\frac{13}{18}\zeta(3) - \frac{5}{144}\pi^2\ln 3 + \frac{1}{48}\ln^3 3 = \mathrm{Re}\,\mathrm{Li}_3\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{6}\right)\right] \tag{47}
$$

and

$$
\begin{aligned}
&\frac{29}{1296}\pi^3 - \frac{1}{48}\pi\ln^2 3 \\
&\quad = 4\,\mathrm{Im}\,\mathrm{Li}_3\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{2}\right)\right] - 5\,\mathrm{Im}\,\mathrm{Li}_3\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{6}\right)\right]\,.
\end{aligned} \tag{48}
$$

Using

$$
\begin{aligned}
\mathrm{Li}_3\left[\pm\frac{1}{3}\right] = \frac{1}{3^6}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}\Bigg[&\pm\frac{3^5}{(6k+1)^3} + \frac{3^4}{(6k+2)^3} \pm \frac{3^3}{(6k+3)^3} \\
&+\frac{3^2}{(6k+4)^3} \pm \frac{3}{(6k+5)^3} + \frac{1}{(6k+6)^3}\Bigg]
\end{aligned} \tag{49}
$$

leads to the ternary BBP-type formula

$$
\begin{aligned}
&13\zeta(3) - \pi^2 \ln 3 + \ln^3 3 \\
&\quad = \frac{2}{3^5} \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \left[\frac{3^6}{(6k+1)^3} + \frac{3^4}{(6k+2)^3} + \frac{3^4}{(6k+3)^3} \right. \\
&\qquad \left. + \frac{3^2}{(6k+4)^3} + \frac{3^2}{(6k+5)^3} + \frac{1}{(6k+6)^3} \right] .
\end{aligned} \tag{50}
$$

A shorter version (length 2) of the above formula is

$$
13\zeta(3) - \pi^2 \ln 3 + \ln^3 3 = \frac{2}{3} \sum_{k=0}^{\infty} \frac{1}{9^k} \left[\frac{9}{(2k+1)^3} + \frac{1}{(2k+2)^3} \right] . \tag{51}
$$

The ternary BBP-type formula that results from Eq. (47) is discussed elsewhere [8].

Next we obtain the ternary BBP-type formula that results from Eq. (48).

$$
\begin{aligned}
\operatorname{Im} \operatorname{Li}_3 \left[\frac{1}{\sqrt{3}} \exp\left(\frac{i\pi}{2} \right) \right] &= \frac{\sqrt{3}}{3^6} \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \left[\frac{3^5}{(12k+1)^3} - \frac{3^4}{(12k+3)^3} \right. \\
&\quad \left. + \frac{3^3}{(12k+5)^3} - \frac{3^2}{(12k+7)^3} + \frac{3}{(12k+9)^3} - \frac{1}{(12k+11)^3} \right]
\end{aligned} \tag{52}
$$

and

$$
\begin{aligned}
\operatorname{Im} \operatorname{Li}_3 \left[\frac{1}{\sqrt{3}} \exp\left(\frac{i\pi}{6} \right) \right] &= \frac{\sqrt{3}}{2 \cdot 3^6} \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \left[\frac{3^5}{(12k+1)^3} + \frac{3^5}{(12k+2)^3} + \frac{2 \cdot 3^4}{(12k+3)^3} \right. \\
&\quad + \frac{3^4}{(12k+4)^3} + \frac{3^3}{(12k+5)^3} - \frac{3^2}{(12k+7)^3} \\
&\quad \left. - \frac{3^2}{(12k+8)^3} - \frac{2 \cdot 3}{(12k+9)^3} - \frac{3}{(12k+10)^3} - \frac{1}{(12k+11)^3} \right]
\end{aligned} \tag{53}
$$

Combining Eqs. (52) and (53) according to the prescription of Eq. (48), we arrive at

$$
\begin{aligned}
\frac{29}{1296}\pi^3 - \frac{1}{48}\pi \ln^2 3 &= \frac{\sqrt{3}}{2 \cdot 3^6} \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \left[\frac{3^6}{(12k+1)^3} - \frac{5 \cdot 3^5}{(12k+2)^3} \right. \\
&\quad - \frac{2 \cdot 3^6}{(12k+3)^3} - \frac{5 \cdot 3^4}{(12k+4)^3} + \frac{3^4}{(12k+5)^3} - \frac{3^3}{(12k+7)^3} \\
&\quad \left. + \frac{5 \cdot 3^2}{(12k+8)^3} + \frac{2 \cdot 3^3}{(12k+9)^3} + \frac{5 \cdot 3}{(12k+10)^3} - \frac{3}{(12k+11)^3} \right] .
\end{aligned} \tag{54}
$$

5 Degree 4 Ternary BBP-type Formulas

A two-variable functional equation for degree 4 polylogarithms (Eq. A.2.7.40 of [6]) reads

$$\begin{aligned}
&\mathrm{Li}_4\left[-\frac{x^2y\eta}{\xi}\right]+\mathrm{Li}_4\left[-\frac{y^2x\xi}{\eta}\right]+\mathrm{Li}_4\left[\frac{x^2y}{\eta^2\xi}\right]+\mathrm{Li}_4\left[\frac{y^2x}{\xi^2\eta}\right]\\
&=6\ \mathrm{Li}_4\left[xy\right]+6\ \mathrm{Li}_4\left[\frac{xy}{\eta\,\xi}\right]+6\ \mathrm{Li}_4\left[-\frac{xy}{\eta}\right]+6\ \mathrm{Li}_4\left[-\frac{xy}{\xi}\right]\\
&+3\ \mathrm{Li}_4\left[x\eta\right]+3\ \mathrm{Li}_4\left[y\xi\right]+3\ \mathrm{Li}_4\left[\frac{x}{\eta}\right]+3\ \mathrm{Li}_4\left[\frac{y}{\xi}\right]+3\ \mathrm{Li}_4\left[-\frac{x\eta}{\xi}\right]\\
&+3\ \mathrm{Li}_4\left[-\frac{y\xi}{\eta}\right]+3\ \mathrm{Li}_4\left[-\frac{x}{\eta\,\xi}\right]+3\ \mathrm{Li}_4\left[-\frac{y}{\eta\,\xi}\right]-6\ \mathrm{Li}_4\left[x\right]\\
&-6\ \mathrm{Li}_4\left[y\right]-6\ \mathrm{Li}_4\left[-\frac{x}{\xi}\right]-6\ \mathrm{Li}_4\left[-\frac{y}{\eta}\right]+3/2\ \ln^2\xi\ln^2\eta\,,
\end{aligned}\tag{55}$$

where $\xi=1-x$, $\eta=1-y$.

Putting $x=-\exp\left(i\pi/3\right)$ and $y=\exp\left(i\pi/3\right)$ in Eq. (55), simplifying and taking real and imaginary parts, we obtain

$$\begin{aligned}
&-12\ \mathrm{Re}\ \mathrm{Li}_4\left[\frac{1}{\sqrt{3}}e^{i\pi/2}\right]-3\ \mathrm{Re}\ \mathrm{Li}_4\left[\frac{1}{\sqrt{3}}e^{i\pi/6}\right]\\
&\quad+\mathrm{Li}_4\left[\frac{1}{3}\right]+\frac{1}{4}\mathrm{Li}_4\left[-\frac{1}{3}\right]\\
&\qquad=-\frac{127\pi^4}{10368}+\frac{1}{64}\pi^2\ln^2 3-\frac{5}{384}\ln^4 3
\end{aligned}\tag{56}$$

and

$$\begin{aligned}
&-12\ \mathrm{Im}\ \mathrm{Li}_4\left[\frac{1}{\sqrt{3}}e^{i\pi/2}\right]+15\ \mathrm{Im}\ \mathrm{Li}_4\left[\frac{1}{\sqrt{3}}e^{i\pi/6}\right]\\
&\quad=\frac{29}{864}\pi^3\ln 3-\frac{1}{96}\pi\ln^3 3-\frac{11}{3}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\,.
\end{aligned}\tag{57}$$

First we proceed to obtain the BBP-type formula invoked by Eq. (56).

Now,

$$\begin{aligned}
\mathrm{Re}\,\mathrm{Li}_4\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{2}\right)\right]&=\frac{1}{3^6}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}\left[-\frac{3^5}{(12k+2)^4}+\frac{3^4}{(12k+4)^4}\right.\\
&\left.-\frac{3^3}{(12k+6)^4}+\frac{3^2}{(12k+8)^4}-\frac{3}{(12k+10)^4}+\frac{1}{(12k+12)^4}\right]
\end{aligned}\tag{58}$$

and

$$\begin{aligned}\operatorname{Re}\operatorname{Li}_4\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{6}\right)\right] = \frac{1}{2\cdot 3^6}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}&\left[\frac{3^6}{(12k+1)^4}+\frac{3^5}{(12k+2)^4}\right.\\&-\frac{3^4}{(12k+4)^4}-\frac{3^4}{(12k+5)^4}-\frac{2\cdot 3^3}{(12k+6)^4}\\&-\frac{3^3}{(12k+7)^4}-\frac{3^2}{(12k+8)^4}+\frac{3}{(12k+10)^4}\\&\left.+\frac{2}{(12k+12)^4}\right].\end{aligned} \tag{59}$$

Also

$$\begin{aligned}\operatorname{Li}_4\left[\pm\frac{1}{3}\right] = \frac{2^4}{3^6}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}&\left[\pm\frac{3^5}{(12k+2)^4}+\frac{3^4}{(12k+4)^4}\right.\\&\pm\frac{3^3}{(12k+6)^4}+\frac{3^2}{(12k+8)^4}\pm\frac{3}{(12k+10)^4}\\&\left.+\frac{1}{(12k+12)^4}\right].\end{aligned} \tag{60}$$

Combining Eqs. (58), (59) and (60) according to Eq. (56), we obtain the following degree 4 ternary BBP-type formula:

$$\begin{aligned}\frac{127\pi^4}{5184}-\frac{\pi^2\ln^2 3}{32}+\frac{5\ln^4 3}{192} = \frac{1}{3^6}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}&\left[\frac{3^7}{(12k+1)^4}-\frac{5\cdot 3^7}{(12k+2)^4}\right.\\&-\frac{19\cdot 3^4}{(12k+4)^4}-\frac{3^5}{(12k+5)^4}-\frac{2\cdot 3^6}{(12k+6)^4}\\&-\frac{3^4}{(12k+7)^4}-\frac{19\cdot 3^2}{(12k+8)^4}-\frac{5\cdot 3^3}{(12k+10)^4}\\&\left.+\frac{3^2}{(12k+11)^4}-\frac{10}{(12k+12)^4}\right].\end{aligned} \tag{61}$$

Next we obtain the BBP-type formula invoked by Eq. (57).

Writing

$$\begin{aligned}&\operatorname{Im}\operatorname{Li}_4\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{2}\right)\right]\\&\quad=\frac{\sqrt{3}}{27}\sum_{k=0}^{\infty}\left(-\frac{1}{27}\right)^k\left[\frac{9}{(6k+1)^4}-\frac{3}{(6k+3)^4}+\frac{1}{(6k+5)^4}\right]\end{aligned} \tag{62}$$

and

$$\operatorname{Im}\operatorname{Li}_4\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{6}\right)\right] = \frac{\sqrt{3}}{54}\sum_{k=0}^{\infty}\left(-\frac{1}{27}\right)^k\left[\frac{9}{(6k+1)^4}+\frac{9}{(6k+2)^4}\right.$$
$$\left.+\frac{6}{(6k+3)^4}+\frac{3}{(6k+4)^4}+\frac{1}{(6k+5)^4}\right], \tag{63}$$

and combining these according to Eq. (57), we obtain the following degree 4 BBP-type formula:

$$11\mathrm{Cl}_4\left(\frac{\pi}{3}\right)-\frac{29}{288}\pi^3\ln 3+\frac{\pi\ln^3 3}{32}$$
$$=\frac{\sqrt{3}}{2}\sum_{k=0}^{\infty}\left(-\frac{1}{27}\right)^k\left[\frac{9}{(6k+1)^4}-\frac{15}{(6k+2)^4}\right.$$
$$\left.-\frac{18}{(6k+3)^4}-\frac{5}{(6k+4)^4}+\frac{1}{(6k+5)^4}\right]. \tag{64}$$

It is interesting to remark that Eq. (64) was also obtained by Broadhurst [9], using the PSLQ Algorithm. We have thus found its formal proof for the first time, through Eq. (57)!

6 Degree 5 Ternary BBP-type Formulas

The following degree 5 polylogarithm identity is derived in [10]

$$\begin{aligned}
&\operatorname{Li}_5\left[\frac{x\alpha}{y\beta}\right]+\operatorname{Li}_5\left[x\alpha\, y\eta\right]+\operatorname{Li}_5\left[\frac{x\alpha\,\beta}{\eta}\right]+\operatorname{Li}_5\left[x\xi\, y\beta\right]+\operatorname{Li}_5\left[\frac{x\xi}{y\eta}\right]\\
&+\operatorname{Li}_5\left[\frac{x\xi\,\eta}{\beta}\right]+\operatorname{Li}_5\left[\frac{\alpha\, y\beta}{\xi}\right]+\operatorname{Li}_5\left[\frac{\alpha}{\xi\, y\eta}\right]+\operatorname{Li}_5\left[\frac{\alpha\,\eta}{\xi\,\beta}\right]\\
&-9\operatorname{Li}_5[xy]-9\operatorname{Li}_5[x\beta]-9\operatorname{Li}_5[x\eta]-9\operatorname{Li}_5\left[\frac{x}{y}\right]-9\operatorname{Li}_5\left[\frac{x}{\beta}\right]\\
&-9\operatorname{Li}_5\left[\frac{x}{\eta}\right]-9\operatorname{Li}_5[\alpha\, y]-9\operatorname{Li}_5[\alpha\,\beta]-9\operatorname{Li}_5[\alpha\,\eta]\\
&-9\operatorname{Li}_5\left[\frac{\alpha}{y}\right]-9\operatorname{Li}_5\left[\frac{\alpha}{\beta}\right]-9\operatorname{Li}_5\left[\frac{\alpha}{\eta}\right]-9\operatorname{Li}_5[\xi\, y]-9\operatorname{Li}_5[\xi\,\beta]\\
&-9\operatorname{Li}_5[\xi\,\eta]-9\operatorname{Li}_5\left[\frac{y}{\xi}\right]-9\operatorname{Li}_5\left[\frac{\beta}{\xi}\right]-9\operatorname{Li}_5\left[\frac{\eta}{\xi}\right]\\
&+18\operatorname{Li}_5[x]+18\operatorname{Li}_5[\alpha]+18\operatorname{Li}_5[\xi]+18\operatorname{Li}_5[y]+18\operatorname{Li}_5[\beta]\\
&+18\operatorname{Li}_5[\eta]-18\,\zeta(5)=3/10\,(\ln\xi)^5+3/4\,(\ln y-\ln x)\,(\ln\xi)^4\\
&+3/2\,(3\,\ln y-\ln\eta)\,(\ln\eta)^2\,(\ln\xi)^2+1/2\,\pi^2\,(\ln\xi-3\,\ln\eta)\,(\ln\xi)^2+1/5\,\pi^4\ln\xi\,.
\end{aligned} \tag{65}$$

Here $\xi = 1-x$, $\eta = 1-y$, $\alpha = -x/\xi$ and $\beta = -y/\eta$.

Putting $x = -\exp(i\pi/3)$ and $y = \exp(i\pi/3)$ in Eq. (65) and simplifying, gives

$$\begin{aligned}&\frac{1}{64}\pi^2\ln^3 3-\frac{127}{3456}\pi^4\ln 3-\frac{1}{128}\ln^5 3+\frac{1573}{144}\zeta(5)\\&=\frac{3}{2}\,\mathrm{Li}_5\left[-\frac{1}{3}\right]-3\,\mathrm{Li}_5\left[\frac{1}{3}\right]+9\,\mathrm{Li}_5\left[\frac{1}{\sqrt{3}}e^{i\pi/6}\right]+9\,\mathrm{Li}_5\left[\frac{1}{\sqrt{3}}e^{-i\pi/6}\right].\end{aligned}\tag{66}$$

On taking real parts

$$\begin{aligned}&\frac{1}{64}\pi^2\ln^3 3-\frac{127}{3456}\pi^4\ln 3-\frac{1}{128}\ln^5 3+\frac{1573}{144}\zeta(5)\\&=18\,\mathrm{Re}\,\mathrm{Li}_5\left[\frac{1}{\sqrt{3}}e^{i\pi/6}\right]+\frac{3}{2}\,\mathrm{Li}_5\left[-\frac{1}{3}\right]-3\,\mathrm{Li}_5\left[\frac{1}{3}\right].\end{aligned}\tag{67}$$

Now

$$\begin{aligned}&18\,\mathrm{Re}\,\mathrm{Li}_5\left[\frac{1}{\sqrt{3}}e^{i\pi/6}\right]=18\sum_{k=0}^{\infty}\left(\frac{1}{\sqrt{3}}\right)^k\frac{1}{k^5}\cos\left(\frac{k\pi}{6}\right)\\&=\frac{1}{3^6}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}\left[\frac{3^8}{(12k+1)^5}+\frac{3^7}{(12k+2)^5}-\frac{3^6}{(12k+4)^5}\right.\\&-\frac{3^6}{(12k+5)^5}-\frac{2\cdot 3^5}{(12k+6)^5}-\frac{3^5}{(12k+7)^5}-\frac{3^4}{(12k+8)^5}\\&\left.+\frac{3^3}{(12k+10)^5}+\frac{3^3}{(12k+11)^5}+\frac{2\cdot 3^2}{(12k+12)^5}\right],\end{aligned}\tag{68}$$

$$\begin{aligned}&\frac{3}{2}\,\mathrm{Li}_5\left[-\frac{1}{3}\right]=\frac{3}{2}\sum_{k=1}^{\infty}\left(-\frac{1}{3}\right)^k\frac{1}{k^5}\\&=\frac{1}{3^6}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}\left[\frac{-3^6\cdot 2^4}{(12k+2)^5}+\frac{3^5\cdot 2^4}{(12k+4)^5}-\frac{3^4\cdot 2^4}{(12k+6)^5}\right.\\&\left.+\frac{3^3\cdot 2^4}{(12k+8)^5}-\frac{3^2\cdot 2^4}{(12k+10)^5}+\frac{3\cdot 2^4}{(12k+12)^5}\right]\end{aligned}\tag{69}$$

and

$$\begin{aligned}&3\,\mathrm{Li}_5\left[\frac{1}{3}\right]=3\sum_{k=1}^{\infty}\left(\frac{1}{3}\right)^k\frac{1}{k^5}\\&=\frac{1}{3^6}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}\left[\frac{3^6\cdot 2^5}{(12k+2)^5}+\frac{3^5\cdot 2^5}{(12k+4)^5}+\frac{3^4\cdot 2^5}{(12k+6)^5}\right.\\&\left.+\frac{3^3\cdot 2^5}{(12k+8)^5}+\frac{3^2\cdot 2^5}{(12k+10)^5}+\frac{3\cdot 2^5}{(12k+12)^5}\right].\end{aligned}\tag{70}$$

Using Eq. (68), (69) and (70) in Eq. (67), we obtain the ternary BBP-type formula

$$\begin{aligned} &\frac{1}{64}\pi^2\ln^3 3-\frac{127}{3456}\pi^4\ln 3-\frac{1}{128}\ln^5 3+\frac{1573}{144}\zeta(5)\\ &=\frac{1}{3^5}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}\left[\frac{3^7}{(12k+1)^5}-\frac{5\cdot 3^7}{(12k+2)^5}-\frac{19\cdot 3^4}{(12k+4)^5}\right.\\ &-\frac{3^5}{(12k+5)^5}-\frac{2\cdot 3^6}{(12k+6)^5}-\frac{3^4}{(12k+7)^5}\\ &\left.-\frac{19\cdot 3^2}{(12k+8)^5}-\frac{5\cdot 3^3}{(12k+10)^5}+\frac{3^2}{(12k+11)^5}-\frac{10}{(12k+12)^5}\right]. \end{aligned} \tag{71}$$

7 Conclusion

Using fairly straightforward methods, we have obtained several ternary BBP-type formulas, which can now be added to the literature. In particular we proved the following formulas (written in the now standard BBP notation [3]).

$\ln 2 = 1/(2\cdot 3^5)\mathrm{P}((1,3^6,12,(3^5,3^5,0,-3^4,-3^3,0,-3^2,-3^2,0,3,1,0)))$

$\ln 3 = 1/3^6\mathrm{P}((1,3^6,12,(3^6,3^5,0,-3^4,-3^4,-2\cdot 3^3,-3^3,-3^2,0,3,3,2)))$

$\ln 3 = 1/27\mathrm{P}((1,-27,6,(27,9,0,-3,-3,-2)))$

$\pi = \sqrt{3}/3^5\mathrm{P}((1,3^6,12,(3^5,3^5,2\cdot 3^4,3^4,3^3,0,-3^2,-3^2,-6,-3,-1,0)))$

$\pi = 2\sqrt{3}/3^5\mathrm{P}((1,3^6,12,(3^5,0,-3^4,0,3^3,0,-3^2,0,3,0,-1,0)))$

$\pi = 2\sqrt{3}\mathrm{P}((1,-3,2,(1,0)))$

$\pi^2 = 2/27\mathrm{P}((2,3^6,12,(3^5,-5\cdot 3^4,0,-3^4,-3^3,-2^3\cdot 3^2,-3^2,-3^2,0,-5,1,0)))$

$\ln^2 3 = 1/3^6\mathrm{P}((2,3^6,12,(2\cdot 3^7,-2\cdot 3^8,0,-2\cdot 13\cdot 3^4,-2\cdot 3^5,-2^3\cdot 3^5,-2\cdot 3^4,$
$-2\cdot 13\cdot 3^2,0,-2\cdot 3^4,2\cdot 3^2,-8)))$

$\pi\ln 3 = 2\sqrt{3}/3\mathrm{P}((2,-27,6,(9,-15,-18,-5,1,0)))$

$\mathrm{Cl}_2(\pi/3) = \sqrt{3}/9\mathrm{P}((2,-27,6,(9,-9,-12,-3,1,0)))$

$13\zeta(3)-\pi^2\ln 3+\ln^3 3 = 2/3^5\mathrm{P}((3,3^6,6,(3^6,3^4,3^4,3^2,3^2,1)))$

$13\zeta(3)-\pi^2\ln 3+\ln^3 3 = 2/3\mathrm{P}((3,9,2,(9,1)))$

$29\pi^3/1296-\pi\ln^2 3/48 = \sqrt{3}/2/3^6\mathrm{P}((3,3^6,12,(3^6,-5\cdot 3^5,$
$-2\cdot 3^6,-5\cdot 3^4,3^4,0,-3^3,5\cdot 3^2,2\cdot 3^3,5\cdot 3,-3,0)))$

$127\pi^4/5184-\pi^2\ln^2 3/32+5\ln^4 3/192 = 1/3^6\mathrm{P}((4,3^6,12,(3^7,-5\cdot 3^7,0,$
$-19\cdot 3^4,-3^5,-2\cdot 3^6,-3^4,-19\cdot 3^2,0,-5\cdot 3^3,3^2,-10)))$

$11\mathrm{Cl}_4(\pi/3) - 29\pi^3 \ln 3/288 + \pi \ln^3 3/32 = \sqrt{3}/2\mathrm{P}((4, -27, 6, (9, -15, -18, -5, 1, 0)))$

$\pi^2 \ln^3 3/64 - 127\pi^4 \ln 3/3456 - \ln^5 3/128 + 1573\zeta(5)/144 = 1/3^5\mathrm{P}((5, 3^6, 12, (3^7,$
$-5 \cdot 3^7, 0, -19 \cdot 3^4, -3^5, -2 \cdot 3^6, -3^4, -19 \cdot 3^2, 0, -5 \cdot 3^3, 3^2, -10)))$

We also proved the following ternary Zero Relations:

$$0 = \mathrm{P}((1, 3^6, 12, (3^5, -5 \cdot 3^4, 0, -3^4, -3^3, -2^3 \cdot 3^2, -3^2, -3^2, 0, -5, 1, 0)))$$

$$0 = \mathrm{P}((1, 3^6, 12, (3^5, -3^5, -2^2 \cdot 3^4, -3^4, 3^3, 0, -3^2, 3^2, 3 \cdot 2^2, 3, -1, 0))).$$

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