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Two modifications of Klamkin's inequality

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Abstract: Two modifications of Klamkin's inequality are formulated and proved.
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In [1] Klamkin introduced the inequality

$$(m+n)(1+x^m) \geq 2n\frac{1-x^{m+n}}{1-x^n}, \tag{1}$$

where $m \geq n \geq 1$ and $x \geq 0, \neq 1$ are real numbers. This inequality is an object of research and application by J. Sandor in [2].

We will modify (1) to two different forms.

Theorem 1. Let $x \geq 0, m \geq k \geq n \geq 1$ and $2k \geq m+n$. Then

$$(m+k+n)(1+x^m)(1+x^k) \geq 3n\frac{1-x^{m+k+n}}{1-x^n}. \tag{2}$$

Proof. Let $x > 1$. Then, (2) has the equivalent forms

$$(m+k+n)(1+x^m)(1+x^k) \geq 3n\frac{x^{m+k+n}-1}{x^n-1} \tag{3}$$

and

$$(m+k+n)(x^m+1)(x^k+1)(x^n-1) \geq 3n(x^{m+k+n}-1). \tag{4}$$

Now, having in mind (1), in its form

$$(m+n)(x^m+1) \geq 2n\frac{x^{m+n}-1}{x^n-1} \tag{5}$$

for (4) we obtain sequentially

$$(m+k+n)(x^m+1)(x^k+1)(x^n-1) - 3n(x^{m+k+n}-1)$$

$$= \frac{m+k+n}{m+n}(m+n)(x^m+1)(x^n-1)(x^k+1) - 3n(x^{m+k+n}-1)$$

$$\geq \frac{m+k+n}{m+n}.2n(x^{m+n}-1)(x^k+1) - 3n(x^{m+k+n}-1)$$

$$= \frac{n}{m+n}(2(m+k+n)(x^{m+k+n} + x^{m+n} - x^k - 1) - 3(m+n)(x^{m+k+n} - 1))$$

$$= \frac{n}{m+n}((2k-m-n)(x^{m+k+n} - 1) + 2(m+k+n)(x^{m+n} - x^k))$$

(from $x^{m+n} \geq x^k$)

$$\geq \frac{n}{m+n}(2k-m-n)(x^{m+k+n} - 1)$$

(from $2k \geq m+n$ and $x > 1$)

$$\geq 0.$$

If $x = 1$, then both sides of (4) are equal to 0. If $x = 0$, then (3) is transformed to the inequality

$$m + k + n \geq 3n,$$

that is true.

Let below $0 < x < 1$. Then, (2) has the form

$$(m+k+n)(1+x^m)(1+x^k)(1-x^n) \geq 3n(1-x^{m+k+n}) \tag{6}$$

Having in mind (1), for (6) we obtain sequentially

$$(m+k+n)(1+x^m)(1+x^k)(1-x^n) - 3n(1-x^{m+k+n})$$

$$= \frac{m+k+n}{m+n}(m+n)(1+x^m)(1-x^n)(1+x^k) - 3n(1-x^{m+k+n})$$

$$\geq \frac{m+k+n}{m+n}.2n(1-x^{m+n})(1+x^k) - 3n(1-x^{m+k+n})$$

$$= \frac{n}{m+n}(2(m+k+n)(1-x^{m+k+n} - x^{m+n} + x^k) - 3(m+n)(1-x^{m+k+n}))$$

$$= \frac{n}{m+n}((2k-m-n)(1-x^{m+k+n}) + 2(m+k+n)(x^k - x^{m+n}))$$

(from $x^{m+n} \leq x^k$)

$$\geq \frac{n}{m+n}(2k-m-n)(1-x^{m+k+n})$$

(from $2k \geq m+n$ and $x < 1$)

$$\geq 0.$$

Therefore, in both cases (2) is valid.

Theorem 2. Let $x \geq 0, k \geq m \geq n \geq 1$ and $m + n \geq k$. Then (2) is valid.

Proof. Let $x > 1$. We will use again (4) (as an equivalent form of (2)) and (5):

$$(k+m+n)(x^k+1)(x^m+1)(x^n-1) - 3n(x^{m+k+n} - 1)$$

$$= \frac{k+m+n}{m+n}(m+n)(x^k+1)(x^n-1)(x^m+1) - 3n(x^{k+m+n} - 1)$$

$$\geq \frac{k+m+n}{m+n}.2n(x^{m+n}-1)(x^k+1) - 3n(x^{k+m+n} - 1)$$

$$= \frac{n}{m+n}(2(m+k+n)(x^{k+m+n} + x^{m+n} - x^k - 1) - 3(m+n)(x^{k+m+n} - 1))$$

$$= \frac{n}{m+n}((2k-m-n)(x^{k+m+n}-1)+2(k+m+n)(x^{m+n}-x^k))$$

(from $x^{m+n} \geq x^k$)

$$\geq \frac{n}{m+n}(2k-m-n)(x^{m+k+n}-1)$$

(from $k \geq m \geq n$ and $x > 1$)

$$\geq 0.$$

The rest of the cases are checked by analogy.

In a next author's research an extension of Klamkin's inequality will be discussed. In it, numbers k, m, n will be changed with s real numbers $m_1, m_2, ..., m_s$.

References

[1] Klamkin, M., Problem E2483, American Mathematical Monthly, Vol. 81, 1974, 660.

[2] Sandor, J., On an inequality of Klamkin, Proceedings of the Jangjeon Mathematical Society, Vol. 13, 2010, No. 1, 49-54.