**NNTDM 16 (2010) 4, 25-28**

# SHARP CONCENTRATION OF THE RAINBOW CONNECTION OF RANDOM GRAPHS

**Yilun Shang**[1,2]
[1]Department of Mathematics, Shanghai Jiao Tong University
Shanghai 200240, China
[2]Institute for Cyber Security, University of Texas at San Antonio
San Antonio, Texas 78249, USA
email: shylmath@hotmail.com

**Abstract:** An edge-colored graph $G$ is rainbow edge-connected if any two vertices are connected by a path whose edges have distinct colors. The rainbow connection of a connected graph $G$, denoted by $rc(G)$, is the smallest number of colors that are needed in order to make $G$ rainbow connected. Similarly, a vertex-colored graph $G$ is rainbow vertex-connected if any two vertices are connected by a path whose internal vertices have distinct colors. The rainbow vertex-connection of a connected graph $G$, denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make $G$ rainbow vertex-connected. We prove that both $rc(G)$ and $rvc(G)$ have sharp concentration in classical random graph model $G(n, p)$.

**Keywords:** Rainbow connection, Graph coloring, Concentration; Random graph.

**AMS Classification Numbers:** 05C80, 05C15, 05C40.

## 1. Introduction

We follow the terminology and notation of [4] in this letter. A natural and interesting connectivity measure of a graph was recently introduced in [6] and has attracted many attention of researchers. An edge-colored graph $G$ is called rainbow edge-connected if any two vertices are connected by a path whose edges have distinct colors. Hence, if a graph is rainbow edge-connected, then it must also be connected. Also notice that any connected graph has a trivial edge coloring that makes it rainbow edge-connected. The rainbow connection of a connected graph $G$, denoted $rc(G)$, is the smallest number of colors that are needed in order to make $G$ rainbow edge-connected.

If $G$ has $n$ vertices then $rc(G) \leq n-1$, since one can color the edges of a given spanning tree of $G$ with distinct colors, and color the remaining edges with one of the already used colors. Obviously, $rc(G) = 1$ if and only if $G$ is a complete graph, and that $rc(G) = n-1$ if and only if $G$ is a tree. An easy observation gives $rc(G) \geq diam(G)$, where $diam(G)$ denotes the diameter of $G$. The behavior of $rc(G)$ with respect to the minimum degree $\delta(G)$ has been addressed in the work [5, 10, 11], which indicate that $rc(G)$ is upper bounded by the reciprocal of $\delta(G)$ up to a multiplicative constant (which we will discuss later). Some related concepts such as rainbow path [9], rainbow tree [8] and rainbow $k$-connectivity [7] have also been investigated recently.

The authors in [10] introduce a vertex coloring edition. A vertex-colored graph $G$ is called rainbow vertex-connected if any two vertices are connected by a path whose internal vertices have distinct colors. Denote the rainbow vertex-connection of a connected graph $G$ by $rvc(G)$, which is defined as the smallest number of colors that are needed in order to make $G$ rainbow vertex-connected. It is clear that $rvcG \leq n-2$, and $rvcG = 0$ if and only if $G$ is complete. Similarly, we have $rvcG \geq diam(G) - 1$.

Note that $rc(G)$ and $rvc(G)$ are both monotonic property in the sense that if we add an edge to $G$ we cannot increase its rainbow edge/vertex-connection. Therefore, it is desirable to study the random graph setting [3]. Motivating this idea, in this paper we consider the rainbow edge/vertex-connection in Erdős-Rényi random graph model $G(n,p)$ with $n$ vertices and edge probability $p \in [0,1]$. Based on some known bounds of diameter and degree of $G(n,p)$, we establish the following concentration results:

**Theorem 1.** *Suppose that $\omega = \omega(n) \to -\infty$ and $c = c(n) \to 0$. Let $d = d(n) \geq 2$ be a natural number and $0 < p = p(n) < 1$. If*

$$np = \ln n + \frac{20n \ln\ln n}{d+1} - \omega, \tag{1}$$

$$p^d n^{d-1} = \ln\left(\frac{n^2}{c}\right) \tag{2}$$

*and*

$$\frac{pn}{(\ln n)^3} \to \infty \tag{3}$$

*hold, then $rc(G(n,p)) = d$ almost surely as $n \to \infty$.*

**Theorem 2.** *Suppose that $\omega = \omega(n) \to -\infty$ and $c = c(n) \to 0$. Let $d = d(n) \geq 2$ be a natural number and $0 < p = p(n) < 1$. If*

$$np = \ln n + \frac{11n \ln\ln n}{d} - \omega, \tag{4}$$

$$p^d n^{d-1} = \ln\left(\frac{n^2}{c}\right) \tag{5}$$

*and*

$$\frac{pn}{(\ln n)^3} \to \infty \tag{6}$$

*hold, then $rvc(G(n,p)) = d-1$ almost surely as $n \to \infty$.*

## 2. Proof of Theorem 1 and 2

In this section, we will first prove Theorem 1 and then Theorem 2 can be derived similarly.

Let $\delta(G)$ be the minimum degree of a graph $G$. The following lemma gives upper bounds of rainbow edge/vertex-connection.

**Lemma 1.**([10]) *A connected graph $G$ with $n$ vertices has $rc(G) < 20n/\delta(G)$ and $rvc(G) < 11n/\delta(G)$.*

**Proof of Theorem 1.** By Lemma 1 and the comments in the Section 1, we have

$$diam(G(n,p)) \leq rc(G(n,p)) < 20n/\delta(G(n,p)) \tag{7}$$

if $G(n,p)$ is connected.

To get the concentration result, we need to estimate the diameter and minimum degree of random graph $G(n,p)$. It follows from the assumptions (2) and (3) that $diam(G(n,p)) = d$ almost surely (see [2] or [3] pp.259). By the assumption (1), we get $\delta(G(n,p)) = 20n/(d+1)$ (see [1] or [3] pp.65). Now we almost conclude our proof by (7).

There are nevertheless two things remain to check: (i) The assumptions (1)-(3) are reasonable, that is, there really exist such $p$ and $d$. (ii) $G(n,p)$ is almost surely connected.

Define $c = c(n) \to 0$ by the equation

$$\ln\ln\left(\frac{n^2}{c}\right) = (\ln n)\cdot\ln\ln n \tag{8}$$

and let $\omega(n) \to -\infty$ sufficiently slowly. By the assumption (1), we define a function of $d$

$$f(d) := (np)^d = \left(\ln n + \frac{20n\ln\ln n}{d+1} - \omega\right)^d. \tag{9}$$

Take $d = \ln n$, and we obtain

$$\begin{aligned}\ln f(d) &= (\ln n)\cdot\ln\left(\ln n + \frac{20n\ln\ln n}{1+\ln n} - \omega\right)\\ &\geq (\ln n)\cdot\ln\left(\frac{n\ln\ln n}{\ln n}\right)\\ &\geq \ln n + (\ln n)\cdot\ln\ln n\\ &= \ln\left(n\cdot\ln\left(\frac{n^2}{c}\right)\right)\end{aligned} \tag{10}$$

where the last equality holds by the definition (8).

Take $d = \ln\ln n$, and we have

$$\begin{aligned}\ln f(d) &= (\ln\ln n)\cdot\ln(\ln n + 20n - \omega)\\ &\leq (\ln\ln n)\cdot\ln(21n)\\ &\leq \ln n + (\ln n)\cdot\ln\ln n\\ &= \ln\left(n\cdot\ln\left(\frac{n^2}{c}\right)\right)\end{aligned} \tag{11}$$

where the last equality holds by the definition (8).

From (10), (11) and the fact that $f(d)$ is continuous, we derive that there exists some $d \in [\ln\ln n, \ln n]$ such that $\ln f(d) = \ln(n\ln(n^2/c))$ holds. Consequently, the assumption (2) holds. For such $d$, by (9), we have

$$np = \Omega\left(\frac{n\ln\ln n}{\ln n}\right), \tag{12}$$

which clearly satisfies the assumption (3), and $G(n,p)$ is connected almost surely (c.f. [3] pp.164).

Hence, both (i) and (ii) have been checked and the proof is finally completed. □

**Proof of Theorem 2.** It can be proved similarly by noting the fact

$$diam(G(n,p)) - 1 \leq rvc(G(n,p)) < 11n/\delta(G(n,p)). \tag{13}$$

We leave the details to the interested readers.□

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