NNTDM 16 (2010) 4, 18-24

COMBINED 2-FIBONACCI SEQUENCES. Part 2

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Abstract. Two new sequences from Fibonacci type are introduced and the explicit formulae for their n-th members are given.

Keywords: Fibonacci sequence, 2-Fibonacci sequence

2000 Mathematics Subject Classification: 11B39

In [2, 4, 5, 6] four different ways of constructing two sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ are described and called *2-Fibonacci sequences* (or *2-F-sequences).* On their base, in [3] the following two new schemes are introduced.

$$\begin{array}{l} \alpha_0 = 2a,\ \beta_0 = 2b,\ \alpha_1 = 2c,\ \beta_1 = 2d \\ \alpha_{n+2} = \dfrac{\alpha_{n+1} + \beta_{n+1}}{2} + \beta_n,\ n \geq 0 \\ \beta_{n+2} = \dfrac{\alpha_{n+1} + \beta_{n+1}}{2} + \alpha_n,\ n \geq 0 \end{array},$$

and

$$\begin{array}{l} \alpha_0 = 2a,\ \beta_0 = 2b,\ \alpha_1 = 2c,\ \beta_1 = 2d \\ \alpha_{n+2} = \dfrac{\alpha_{n+1} + \beta_{n+1}}{2} + \alpha_n,\ n \geq 0 \\ \beta_{n+2} = \dfrac{\alpha_{n+1} + \beta_{n+1}}{2} + \beta_n,\ n \geq 0 \end{array}.$$

Let σ be the integer function defined for every $k \geq 0$ by:

r	$\sigma(4.k + r)$
0	0
1	1
2	0
3	-1

Obviously, for every $n \geq 0$,

$$\sigma(n+2) + \sigma(n) = 0.$$

In [3] the following two assertions are formulated and proved for these two sequences.

THEOREM 1. For every natural number $n \geq 0$

$$\alpha_{n+2} = (F_{n+1} + \sigma(n-1)).a + (F_{n+1} + \sigma(n+1)).b + (F_{n+2} + \sigma(n+2)).c + (F_{n+2} + \sigma(n)).d$$

$$\beta_{n+2} = (F_{n+1} + \sigma(n+1)).a + (F_{n+1} + \sigma(n-1)).b + (F_{n+2} + \sigma(n)).c + (F_{n+2} + \sigma(n+2)).d.$$

THEOREM 2. For each natural number $n \geq 0$

$$\alpha_{n+2} = (F_{n+1} + \rho(n)).a + (F_{n+1} - \rho(n)).b + (F_{n+2} + \rho(n+1)).c + (F_{n+2} - \rho(n+1)).d$$

$$\beta_{n+2} = (F_{n+1} - \rho(n)).a + (F_{n+1} + \rho(n)).b + (F_{n+2} - \rho(n+1)).c + (F_{n+2} + \rho(n+1)).d.$$

Now, we will introduce two new schemes. The first one is:

$$\alpha_0 = 2a, \ \beta_0 = 2b, \ \alpha_1 = 2c, \ \beta_1 = 2d$$

$$\alpha_{n+2} = \beta_{n+1} + \frac{\alpha_n + \beta_n}{2}, \ n \geq 0 \qquad ,$$

$$\beta_{n+2} = \alpha_{n+1} + \frac{\alpha_n + \beta_n}{2}, \ n \geq 0$$

where a, b, c, d are given constants.

If we set $a = b$ and $c = d$, then sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ will coincide with each other and with the sequence $\{F_i\}_{i=0}^{\infty}$, which is called a generalized Fibonacci sequence, where

$$F_0(a,c) = a,$$

$$F_1(a,c) = c,$$

$$F_{n+2}(a,c) = F_{n+1}(a,c) + F_n(a,c).$$

Let $F_i = F_i(0,1)$; $\{F_i\}_{i=0}^{\infty}$ be the ordinary Fibonacci sequence.

The first 10 members of the first of the new schemes have the form shown on Table 1.

Table 1

	α_n	β_n
0	$2a$	$2b$
1	$2c$	$2d$
2	$a+b+2d$	$a+b+2c$
3	$a+b+3c+d$	$a+b+c+3d$
4	$2a+2b+2c+4d$	$2a+2b+4c+2d$

5	$3a+3b+6c+4d$	$3a+3b+4c+6d$
6	$5a+5b+7c+9d$	$5a+5b+9c+7d$
7	$8a+8b+14c+12d$	$8a+8b+12c+14d$
8	$13a+13b+29c+22d$	$13a+13b+22c+20d$
9	$21a+21b+35c+33d$	$21a+21b+33c+35d$

THEOREM 3. For every natural number $n \geq 0$

$$\alpha_{n+2} = F_{n+1}.a + F_{n+1}.b + (F_{n+2} + (-1)^{n+1}).c + (F_{n+2} + (-1)^{n}).d$$

$$\beta_{n+2} = F_{n+1}.a + F_{n+1}.b + (F_{n+2} + (-1)^{n}).c + (F_{n+2} + (-1)^{n+1}).d$$

The proof of this assertion can be made, for example, by induction.

For $n = 0$ we see the validity of the two formulas from Table 1. Let us assume that these formulas are valid for some natural number $n \geq 0$. Then, having in mond that for every natural number $n \geq 0$

$$(-1)^n + (-1)^{n+1} = 0,$$

we obtain

$$\alpha_{n+3} = \beta_{n+2} + \frac{\alpha_{n+1} + \beta_{n+1}}{2}$$

$$= F_{n+1}.a + F_{n+1}.b + (F_{n+2} + (-1)^{n}).c + (F_{n+2} + (-1)^{n+1}).d$$

$$+\frac{1}{2}.(F_n.a + F_n.b + (F_{n+1} + (-1)^{n}).c + (F_n + (-1)^{n-1}).d$$

$$+F_n.a + F_n.b + (F_{n+1} + (-1)^{n-1}).c + (F_{n+1} + (-1)^{n}).d))$$

$$= F_{n+1}.a + F_{n+1}.b + (F_{n+2} + (-1)^{n}).c + (F_{n+2} + (-1)^{n+1}).d$$

$$+F_n.a + F_n.b + F_{n+1}.c + F_n.d$$

$$= F_{n+2}.a + F_{n+2}.b + (F_{n+3} + (-1)^{n}).c + (F_{n+3} + (-1)^{n+1}).d$$

$$= F_{n+2}.a + F_{n+2}.b + (F_{n+3} + (-1)^{n+2}).c + (F_{n+3} + (-1)^{n+1}).d.$$

The formula for β_{n+3} may be checked in similar manner.

The second new sequence has the form:

$$\alpha_0 = 2a,\ \beta_0 = 2b,\ \alpha_1 = 2c,\ \beta_1 = 2d$$

$$\alpha_{n+2} = \alpha_{n+1} + \frac{\alpha_n + \beta_n}{2},\ n \geq 0 \qquad ,$$

$$\beta_{n+2} = \beta_{n+1} + \frac{\alpha_n + \beta_n}{2},\ n \geq 0$$

where a, b, c, d are given constants.

The first 10 members of the second of the new schemes have the form shown on Table 2.

Table 2

	α_n	β_n
0	$2a$	$2b$
1	$2c$	$2d$
2	$a+b+2c$	$a+b+2d$
3	$a+b+3c+d$	$a+b+c+3d$
4	$2a+2b+4c+2d$	$2a+2b+2c+4d$
5	$3a+3b+6c+4d$	$3a+3b+4c+6d$
6	$5a+5b+9c+7d$	$5a+5b+7c+9d$
7	$8a+8b+14c+12d$	$8a+8b+12c+14d$
8	$13a+13b+22c+20d$	$13a+13b+20c+22d$
9	$21a+21b+35c+33d$	$21a+21b+33c+35d$

THEOREM 4. For each natural number $n \geq 0$

$$\alpha_{n+2} = F_{n+1}.a + F_{n+1}.b + (F_{n+2}+1).c + (F_{n+2}-1).d$$

$$\beta_{n+2} = F_{n+1}.a + F_{n+1}.b + +(F_{n+2}-1).c + (F_{n+2}+1).d.$$

For $n = 0$ we see the validity of the two formulas from Table 2. Let us assume that these formulas are valid for some natural number $n \geq 0$. We shall check the validity of the second formula for $n+1$.

$$\beta_{n+3} = \beta_{n+2} + \frac{\alpha_{n+1}+\beta_{n+1}}{2}$$

$$= F_{n+1}.a + F_{n+1}.b + (F_{n+2}-1).c + (F_{n+2}+1).d$$

$$+\frac{1}{2}(F_n.a + F_n.b + (F_{n+1}+1).c + (F_{n+1}-1).d$$

$$(F_n.a + F_n.b + (F_{n+1}-1).c + (F_{n+1}+1).d)$$

$$= F_{n+1}.a + F_{n+1}.b + (F_{n+2}-1).c + (F_{n+2}+1).d$$

$$+F_n.a + F_n.b + F_{n+1}c + F_{n+1}.d$$

$$= F_{n+1}.a + F_{n+1}.b + (F_{n+3}-1).c + (F_{n+3}+1).d.$$

The formula for α_{n+3} may be checked in similar manner.

2. A digital arithmetic function will be described, following [1, 7].
Let

$$n = \sum_{i=1}^{k} a_i.10^{k-i} \equiv \overline{a_1 a_2 ... a_k},$$

where a_i is a natural number and $0 \leq a_i \leq 9$ $(1 \leq i \leq k)$. Let for $n = 0 : \varphi(n) = 0$ and for $n > 0$:

$$\varphi(n) = \sum_{i=1}^{k} a_i.$$

We shall use the decimal count system everywhere hereafter.

Let us define a sequence of functions $\varphi_0, \varphi_1, \varphi_2, ...$, where ($l$ is a natural number)

$$\varphi_0(n) = n,$$

$$\varphi_{l+1} = \varphi(\varphi_l(n)).$$

Obviously, for every $l \in \mathcal{N}$: $\varphi_l : \mathcal{N} \rightarrow \mathcal{N}$. Since for $k > 1$

$$\varphi(n) = \sum_{i=1}^{k} a_i < \sum_{i=1}^{k} a_i.10^{k-i} = n.$$

Then for every $n \in \mathcal{N}$, $l \in \mathcal{N}$ will exist so that

$$\varphi_l(n) = \varphi_{l+1}(n) \in \Delta \equiv \{0, 1, 2, ..., 9\}.$$

Let function ψ be defined by

$$\psi(n) = \varphi_l(n),$$

where

$$\varphi_{l+1}(n) = \varphi_l(n).$$

Let be given the sequence $a_1, a_2, ...$, with its members being natural numbers and let

$$c_i = \psi(a_i) \ (i = 1, 2, ...).$$

Hence, we deduce the sequence $c_1, c_2, ...$ from the former sequence. If k and l exist so that $l \geq 0$,

$$c_{i+l} = c_{k+i+l} = c_{2k+i+l} = ...$$

for $1 \leq i \leq k$, then we, following [1], shall say that $[c_{l+1}, c_{l+2}, ..., c_{l+k}]$ is *base* of the sequence $a_1, a_2, ...$ with length of k and with respect to function ψ.

On Tables 3 and 4 we shall show that the two new sequences have bases with lenght 24.

Table 3

	$\psi(\alpha_n) = \psi(\bullet)$	$\psi(\beta_n) = \psi(\bullet)$
0	$2a$	$2b$
1	$2c$	$2d$
2	$a + b + 2d$	$a + b + 2c$
3	$a + b + 3c + d$	$a + b + c + 3d$

4	$2a+2b+2c+4d$	$2a+2b+4c+2d$
5	$3a+3b+6c+4d$	$3a+3b+4c+6d$
6	$5a+5b+7c$	$5a+5b+7d$
7	$8a+8b+5c+3d$	$8a+8b+3c+5d$
8	$4a+4b+2c+4d$	$4a+4b+4c+2d$
9	$3a+3b+8c+6d$	$3a+3b+6c+8d$
10	$7a+7b+2d$	$7a+7b+2c$
11	$a+b+7d$	$a+b+7c$
12	$8a+8b+8c+d$	$8a+8b+c+8d$
13	$7d$	$7c$
14	$8a+8b+7c$	$8a+8b+7d$
15	$8a+8b+8c+6d$	$8a+8b+6c+8d$
16	$7a+7b+5c+7d$	$7a+7b+7c+5d$
17	$6a+6b+5c+3d$	$6a+6b+3c+5d$
18	$4a+4b++2d$	$4a+4b+2c$
19	$a+b+6c+4d$	$a+b+4c+6d$
20	$5a+5b+5c+7d$	$5a+5b+7c+5d$
21	$6a+6b+3c+d$	$6a+6b+c+3d$
22	$2a+2b+7c$	$2a+2b++7d$
23	$8a+8b+2c$	$8a+8b+2d$
24	$a+b+8c+d$	$a+b+c+8d$
25	$2c$	$2d$

Table 4

	$\psi(\alpha_n)=\psi(\bullet)$	$\psi(\beta_n)=\psi(\bullet)$
0	$2a$	$2b$
1	$2c$	$2d$
2	$a+b+2c$	$a+b+2d$
3	$a+b+3c+d$	$a+b+c+3d$
4	$2a+2b+4c+2d$	$2a+2b+2c+4d$
5	$3a+3b+6c+4d$	$3a+3b+4c+6d$
6	$5a+5b+7d$	$5a+5b+7c$
7	$8a+8b+5c+3d$	$8a+8b+3c+5d$
8	$4a+4b+4c+2d$	$4a+4b+2c+4d$
9	$3a+3b+8c+6d$	$3a+3b+6c+8d$
10	$7a+7b+2c$	$7a+7b+2d$
11	$a+b+7d$	$a+b+7c$

12	$8a + 8b + c + 8d$	$8a + 8b + 8c + d$
13	$7d$	$7c$
14	$8a + 8b + 7d$	$8a + 8b + 7c$
15	$8a + 8b + 8c + 6d$	$8a + 8b + 6c + 8d$
16	$7a + 7b + 7c + 5d$	$7a + 7b + 5c + 7d$
17	$6a + 6b + 5c + 3d$	$6a + 6b + 3c + 5d$
18	$4a + 4b + 2c$	$4a + 4b + 2d$
19	$a + b + 6c + 4d$	$a + b + 4c + 6d$
20	$5a + 5b + 7c + 5d$	$5a + 5b + 5c + 7d$
21	$6a + 6b + 3c + d$	$6a + 6b + c + 3d$
22	$2a + 2b + 7d$	$2a + 2b + 7c$
23	$8a + 8b + 2c$	$8a + 8b + 2d$
24	$a + b + c + 8d$	$a + b + 8c + d$
25	$2c$	$2d$

References

[1] Atanassov, K. An arithmetic function and some of its applications. Bull. of Number Theory and Related Topics, Vol. IX (1985), No. 1, 18-27.

[2] Atanassov K., On a second new generalization of the Fibonacci sequence. The Fibonacci Quarterly, Vol. 24 (1986), No. 4, 362-365.

[3] Atanassov, K., Combined 2-Fibonacci sequences, Notes on Number Theory and Discrete Mathematics, Vol. 16, 2010, No. 1, 24-28.

[4] Atanassov K., L. Atanassova, D. Sasselov, A new perspective to the generalization of the Fibonacci sequence, The Fibonacci Quarterly, Vol. 23 (1985), No. 1, 21-28.

[5] Atanassov K., V. Atanassova, A. Shannon, J. Turner, New Visual Perspectives on Fibonacci Numbers. World Scientific, New Jersey, 2002.

[6] Lee J.-Z., J.-S. Lee, Some properties of the generalization of the Fibonacci sequence. The Fibonacci Quarterly, Vol. 25 (1987) No. 2, 111-117.

[7] Shannon A., R. Melham, Carlitz generalizations of Lucas and Lehmer sequences, The Fibonacci Quartarly, Vol. 31 (1993), No. 2, 105-111.