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ON THE DIOPHANTINE EQUATION $\sum_{i=1}^{k} \frac{1}{x_i} = 1$ IN DISTINCT INTEGERS OF THE FORM $x_i \in p^{\alpha}q^{\beta}$

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Abstract: A complete demonstration of solutions of the above Diophantine equation is given when $p < q$ are primes and α, β are positive integers. Among the several examples exhibited, Example 3 provides a new solution containing eighty-five even numbers x_i all of which are of the required form. Certain questions and modifications of the equation are also discussed.

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In this paper we consider the Diophantine equation in integers

$$\sum_{i=1}^{k} \frac{1}{x_i} = 1, \quad x_1 < x_2 < \ldots < x_k \tag{1}$$

with $x_i = p^{\alpha}q^{\beta}$ when $p < q$ are primes, and α, β are positive integers.

Two cases will be discussed. **Case 1** when all values x_i are $x_i = pq$, and **Case 2** when not all values x_i are $x_i = pq$.

In order to provide a more comprehensive demonstration of the title equation, we exhibit here Examples 1, 2 already obtained by the author in [2, 5].

Case 1. All values x_i in (1) are $x_i = pq$ $(\alpha = \beta = 1)$.
The above restriction imposed on the values x_i immediately implies the fact that $x_i \nmid x_j$ for $i \neq j$. However, the converse of this statement is false. For if $x_i \nmid x_j$ for $i \neq j$ is true for all values x_i in (1), then all values x_i need not be of the form $x_i = pq$. This is shown in [1, Example 4]. The solution of this case, namely Example 1 is cited in [5], and is the best known result thus far.

Example 1. The 52 different numbers in Table 1. have the following two properties :

(i) each number is a product of two distinct primes, and therefore no-one divides any other,

(ii) the sum of their reciprocals is equal to 1.

6	10	14	15	21	22	26	33	34	35	38
39	46	51	55	57	58	62	65	69	77	82
87	91	93	95	106	119	122	123	133	155	159
161	183	187	202	203	213	265	287	299	319	355
453	497	505	583	671	1057	1313	1963			

Table 1.

Remark 1. The affirmative answer to (1) with the restriction that $x_i \nmid x_j$ for $i \neq j$ raises several questions. Some of these are mentioned in [1, Questions 5, 9], and are respectively as follows:

- Does (1) have a solution with integers $x_1, x_2, \ldots, x_k$ for large x_1, which also satisfy $x_i \nmid x_j$ for $i \neq j$?
- Does (1) have a solution with even integers $x_1, x_2, \ldots, x_k$ which also satisfy $x_i \nmid x_j$ for $i \neq j$?

We mention here that the open Questions 5 and 9 in [1] are still unanswered.

Case 2. All values x_i in (1) are $x_i = p^\alpha q^\beta$.
Allan Johnson [6] found a solution of (1) using $k = 48$ distinct integers all of which are of the form $x_i = p^\alpha q^\beta$. The author [2] improved Johnson's result to $k = 25$. The solution in [2] is presented as Example 2.

Example 2. The 25 different numbers in Table 2. have the following two properties:
(i) each number is of the form $p^\alpha q^\beta$,
(ii) the sum of their reciprocals is equal to 1.

6	10	12	14	15	18	21	22	24
26	28	33	35	36	39	48	52	56
65	72	88	91	99	117	144		

Table 2.

We note that Example 2 is composed of odd and even numbers. This leads us to examine solutions of (1) when all x_i are odd, and when all x_i are even numbers. It is mentioned [6], that the case of all x_i are odd numbers cannot be applied to (1). However, in Example 3, we exhibit a solution of (1) in which all x_i are even numbers.

Example 3. The eighty-five different numbers contained in **(a_1)**, **(a_2)**, **(a_3)**, **(a_4)**, **(b)**, **(c_1)**, **(c_2)** and **(d)** from Table 3 have the following two properties:

- (i) each number is even and of the form $p^\alpha q^\beta$,
- (ii) the sum of their reciprocals is equal to 1.

Clearly this can be verified by means of a computer (L.C.M. = 417533276160), but it can also be done directly by observing the following. Consider the two sets of numbers:

$$S = \{2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096\},$$

$$T = \{3, 7, 3^2, 13, 5, 11, 31, 73\}.$$

The numbers in S are the first twelve powers of 2, and the sets S, T are disjoint sets. Multiplying the twelve numbers in S by each of 3, 7, 3^2 and 13 in T, respectively results in **(a_1)**, **(a_2)**, **(a_3)**, **(a_4)**. The numbers in **(b)** are products of the first eight numbers in S by 5. Multiplying the first ten numbers in S by 11 and 31 in T, respectively yields **(c_1)** and **(c_2)**. Finally, the numbers in **(d)** are products of the first nine numbers in S by 73.

Due to the property of S, T, and to the form just described of the numbers in **(a_1)**, **(a_2), (a_3), (a_4), (b)**, **(c_1), (c_2)** and **(d)**, it is evident that property (i) follows.

For (ii) observe that: All the multiples appearing in Example 3 of a certain number N, where $N \in T$ occur in one and only one row of **(a_1)** through **(d)**. Computing the sum of the reciprocals in each row, we obtain a fraction, the numerator of which is also a multiple of N. After simplification, the new fraction will have a denominator which is a divisor of 4096. This enables us to carry out the summation without a computer.

The eight partial sums add up to 1.

In the following Remark 2 we apply some modifications to (1).

Remark 2. Many problems are concerned with (1), and in particular when all x_i are odd numbers, thus removing the restriction on $x_i = p^\alpha q^\beta$. In this case, it is already known [3] that k must be an odd value. With fixed values k, fixed primes, more than two primes occur, and allowing certain exponents to be zero, it is established :

- (i) For odd values x_i, it is shown [3] that the smallest possible value k is $k = 9$, and for that value k (1) has exactly five solutions with $x_i \in 3^\alpha \cdot 5 \cdot 7 \cdot 11$ where $\alpha \le 3$.
- (ii) It is easily verified that (1) has no solutions when $x_i \in 3^\alpha 5^\beta$. Therefore, when $k =$ 11 and with the three smallest primes, namely $x_i \in 3^\alpha 5^\beta 7^\gamma$, it is known [4] that (1) has exactly seventeen solutions.

(a_1)	6	12	24	48	96	192	384	768	1536	3072	6144	12288
(a_2)	14	28	56	112	224	448	896	1792	3584	7168	14336	28672
(a_3)	18	36	72	144	288	576	1152	2304	4608	9216	18432	36864
(a_4)	24	52	104	208	416	832	1664	3328	6656	13312	26624	53248
(b)	10	20	40	80	160	320	640	1280				
(c_1)	22	44	88	176	352	704	1408	2816	5632	11264		
(c_2)	62	124	248	496	992	1984	3968	7936	15872	31744		
(d)	146	292	584	1168	2336	4672	9344	18688	37376			

Table 3.

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