NNTDM 16 (2010) 3, 18-24

SOME RESULTS ON INFINITE POWER TOWERS

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To my friend Krastyu Gumnerov

Abstract

In the paper the infinite power towers which are generated by an algebraic numbers belonging to the closed interval $[1, e^{\frac{1}{e}}]$ are investigated and an answer is given to the question when they are transcendental or rational numbers. Also a necessary condition for an infinite power tower to be an irrational algebraic number is proposed.

Keywords: *Infinite power tower, Algebraic number, Transcendental number*

Below the following variant of Gelfond-Schneider theorem (see[1]) shall be used:

Theorem 1. *If a $(a \neq 0, 1)$ is an algebraic number and b is an irrational algebraic number, then a^b is a transcendental number.*

Further we shall use the denotation $\sqrt[x]{x}$ for $x^{\frac{1}{x}}$ (where $x^{\frac{1}{x}} \stackrel{\text{def}}{=} e^{\frac{\ln x}{x}}$) and as usual e for John Napier's number $e = 2.71828....$ Let us note that $\sqrt[e]{e} = 1.44466...$

First we need the following:

Lemma 1. *Let for every real $x \geq 1$*

$$f(x) = \sqrt[x]{x}.$$

Then the function f has the following properties:

(a) $f : (1, +\infty) \to (1, \sqrt[e]{e})$

(b) $f(1) = 1$ *;* $f(e) = \sqrt[e]{e}$*;* $\lim\limits_{x \to +\infty} f(x) = 1$

(c) *f is a continuous function strictly increasing on the interval $(1, e)$ and strictly decreasing on the interval $(e, +\infty)$ and f has an absolute maximum at $x = e$, i.e. for $x \in (1, e)$ we have:*

$$1 = f(1) < f(x) < f(e) = \sqrt[e]{e} \tag{1}$$

and for $x \in (e, +\infty)$ we have

$$\sqrt[e]{e} = f(e) > f(x) > 1 = \lim_{x\to\infty} f(x).$$

We omit the elementary proof of this lemma.

As an obvious corollary of the above lemma we obtain:

Lemma 2. *Let $a \in (1, +\infty)$ be a real number. Then:*

(a) *For $a \in (1, \sqrt[e]{e})$ the equation*

$$\sqrt[x]{x} = a \tag{2}$$

has exactly two different solutions $x_1 \in (1, e)$ and $x_2 \in (e, +\infty)$.

(b) *For $a = \sqrt[e]{e}$ the equation (2) has the unique solution $x = e$*

(c) *For $a > \sqrt[e]{e}$ the equation (2) has no solution.*

Our first important result in the paper is (see also [2, Theorem 2]):

Theorem 2. *Let $a \in (1, \sqrt[e]{e})$ be an algebraic number that cannot be represented in the form*

$$a = \sqrt[b]{b} \tag{3}$$

for any rational number $b > 1$. Then:

(a) *The equation*

$$a^x = x \tag{4}$$

has exactly two different solutions: $x_1 \in (1, e)$ and $x_2 \in (e, +\infty)$;

(b) *x_i $(i = 1, 2)$ are transcendental numbers;*

(c) *The algebraic number a admits the following two different representations, using the transcendental numbers x_1 and x_2:*

$$a = \sqrt[x_1]{x_1} \quad \text{and} \quad a = \sqrt[x_2]{x_2}. \tag{5}$$

Proof. We note that (4) is equivalent to (2). Hence **(a)** immediately follows from Lemma 2 **(a)**.

Let $x = b$ be any solution of (4). Then $x = b$ is a solution of (2) too. Therefore, b satisfies (3). Hence b is an irrational number because of the condition of the theorem. Let us assume that b is an algebraic number. Then Theorem 1 yields that a^b is a transcendental number. But

$$a^b = b,$$

since $x = b$ is a solution of(4). Hence b is a transcendental number too. The last contradicts to the assumption that b is an algebraic number. Therefore, our assumption that b is an algebraic number is wrong. Hence b is a transcendental number and **(b)** is proved.

Now, **(c)** (in particular(5)) holds from **(a)** and **(b)**.

The Theorem is proved. □

Remark 1. *If $a > 1$ is an algebraic number given by* (3)*, then either b is a rational number or b is a transcendental number.*

Indeed, if we assume that b is an irrational algebraic number, then according to Theorem 1 $\sqrt[b]{b}$ is a transcendental number which means that a is a transcendental number (because of (3)) in contradiction to the fact that a is an algebraic number.

Let $a \geq 1$ be a real number. Then we consider an infinite sequence $\{K_n(a)\}_{n=1}^{\infty}$ given by

$$K_1(a) = a, \quad K_{n+1}(a) = a^{K_n(a)}, \text{ for } n \geq 1 \tag{6}$$

Definition. *If there exists $\lim_{n\to\infty} K_n(a)$ we denote it by $K(a)$, i.e*

$$K(a) \stackrel{\text{def}}{=} a^{a^{a^{\cdot^{\cdot^{\cdot}}}}}$$

and we call $K(a)$ infinite power tower generated by a.

Let us suppose that for a given $a \geq 1$, $K(a)$ exists. Then putting

$$K(a) = x,$$

from (6) after passage to the limit, we obtain:

$$x = a^x.$$

(i.e. (4))

Hence,

$$a = \sqrt[x]{x}$$

(i.e. (2))

and from (1) we obtain

$$a = \sqrt[x]{x} \leq \sqrt[e]{e}$$

Therefore, using Lemma 2 **(c)**, we get:

Lemma 3. *Let $a \geq 1$ be a real number. Then the necessary condition for the existence of $K(a)$ is $a \in [1, \sqrt[e]{e}]$.*

Lemma 3 yields

Corollary 1. *For $a > \sqrt[e]{e}$ the infinite power tower $K(a)$ does not exist.*

In [2], with the help of Theorem 1, the following result is established:

Theorem 3. *Let $a \in (1, \sqrt[e]{e})$ be a real number. Then the infinite power tower $K(a)$ exists, $K(a)$ belongs to $(1, e)$ and $x = K(a)$ satisfies the equation (2). If a satisfies the conditions of Theorem 2, then $K(a)$ is a transcendental number.*

Remark 2. *We note that $K(1) = 1$ and $K(\sqrt[e]{e}) = e$.*

Thus, the question that remains to be answered is what happens when $a \in (1, \sqrt[e]{e})$ is an algebraic number which admits the representation

$$a = \sqrt[b]{b},$$

where $b > 1$ is a rational number. In this case, from (4) we obtain:

$$\sqrt[x]{x} = \sqrt[b]{b} \tag{7}$$

Further we will consider the following two cases:

Case 1 $b \in (1, e)$.

Case 2 $b \in (e, +\infty)$.

Let **Case 1** hold. The following considerations are valid not only for the case when b is a rational number but also when b is an arbitrary real number. In this case, from (1), (3) , Lemma 2 **(a)** and Theorem 3, it follows that

$$K(a) = \sqrt[b]{b}^{\sqrt[b]{b}^{\sqrt[b]{b}^{\cdot^{\cdot^{\cdot}}}}} = b \tag{8}$$

From (8) it is seen that in **Case 1** $K(a)$ coincides with the rational number b.

Let **Case 2** hold. In this case we have to consider two possibilities for $x = K(a)$:

(i) x is a rational number belonging to $(1, e)$.

(ii) x is an irrational number belonging to $(1, e)$.

Let **(i)** hold. Then the equation (7) is satisfied with rational number $x \in (1, e)$ and rational number $b \in (e, +\infty)$. According to [3, problem 124, p.28] all such rational solutions of (7) are given by:

$$x = \left(1 + \frac{1}{s}\right)^{s}; \quad b = \left(1 + \frac{1}{s}\right)^{s+1}$$

$s = 1, 2, 3, \ldots$

Therefore, each one of them is obtained for an appropriate integer $s \geq 1$. In this case

$$a = \sqrt[(1+\frac{1}{s})^s]{\left(1 + \frac{1}{s}\right)^s}$$

and

$$K(a) = \left(1 + \frac{1}{s}\right)^s$$

is a rational number.

If $b \in (e, +\infty)$ is a rational number that does not belong to the infinite sequence $\left\{\left(1 + \frac{1}{s}\right)^{s+1}\right\}_{s=1}^{\infty}$, then x, that satisfies (7), is not a rational number. Therefore, x satisfies **(ii)**.

Let **(ii)** be fulfilled. Then it follows that x is a transcendental number. Indeed, if we assume that x is an irrational algebraic number, then according to Theorem 1 a^x is a transcendental number and since $a^x = x$, x is a transcendental number too, which contradicts to the assumption that x is an algebraic number.

So when **(ii)** holds, $K(a)$ is a transcendental number. Thus we proved the following

Theorem 4. *Let $b > 1$ be a rational number and $a = \sqrt[b]{b}$. Then:*

a) *If $b \in (1, e)$, then the infinite power tower $K(a)$ is a rational number, and moreover*

$$K(a) = b$$

b) *If $b = \left(1 + \frac{1}{s}\right)^{s+1}$ for some integer $s \geq 1$, then we have $b \in (e, +\infty)$, $a = \sqrt[(1+\frac{1}{s})^s]{\left(1 + \frac{1}{s}\right)^s}$ and the infinite power tower $K(a)$ is a rational number given by:*

$$K(a) = \left(1 + \frac{1}{s}\right)^s$$

c) *If $b \in (e, +\infty)$ and b is not a term of the sequence $\left\{\left(1 + \frac{1}{s}\right)^{s+1}\right\}_{s=1}^{\infty}$, then the infinite power tower $K(a)$ is a transcendental number.*

Now by combining the results from Theorem 3 and Theorem 4 we are ready to formulate the main result of the paper which gives us the answer what is the nature of the infinite power tower K(a) when a $\in (1, \sqrt[e]{e})$ is an algebraic number.

Theorem 5. *Let $a \in (1, \sqrt[e]{e})$ be an algebraic number. Then:*

(a) *If $a \neq \sqrt[b]{b}$ for every rational $b > 1$, then the infinite power tower $K(a)$ is a transcendental number.*

(b) *If $a = \sqrt[b]{b}$ for some rational number $b > 1$, then:*

($\mathbf{b_1}$) *if $b \in (1, e)$, then the infinite power tower $K(a)$ is the rational number b;*

($\mathbf{b_2}$) *if $b \in (e, +\infty)$, then*

($\mathbf{b_{21}}$) *if $b = \left(1+\frac{1}{s}\right)^{s+1}$ for some integer $s \geq 1$, then the infinite power tower $K(a)$ is the rational number $\left(1+\frac{1}{s}\right)^{s}$;*

($\mathbf{b_{22}}$) *if b is not a term of the sequence $\left\{\left(1+\frac{1}{s}\right)^{s+1}\right\}_{s=1}^{\infty}$, then the infinite power tower $K(a)$ is a transcendental number.*

Remark 3. *Since $K(1) = 1$, the infinite power tower $K(1)$ generated by 1 is the rational number 1. Since, $K(\sqrt[e]{e}) = e$, (see Remark 2) and e is a transcendental number, the infinite power tower $K(\sqrt[e]{e}) = e$ is a transcendental number.*

Let $a \in (1, \sqrt[e]{e})$ be a transcendental number (the case which is not investigated in Theorem 5). Then we put $K(a) = x$ and the equality (2) yields that x is not a rational number. Therefore, in this case the infinite power tower $K(a)$ is an irrational algebraic number or a transcendental number.

Thus we obtain

Corollary 2. *A necessary condition for the infinite power tower $K(a)$ to be irrational algebraic number is $a \in (1, \sqrt[e]{e}]$ to be a transcendental number.*

Remark 4. ***Thus we see that if a is an algebraic number then the infinite power tower $K(a)$ can not be irrational algebraic number and if a is a transcendental number then the infinite power tower $K(a)$ cannot be a rational number.***

As a corollary from the results in the paper we obtain that

$$\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot}}}}} = \sqrt[4]{4}^{\sqrt[4]{4}^{\sqrt[4]{4}^{\cdot^{\cdot^{\cdot}}}}} = 2$$

and for every integer n different from $1, 2$ and 4 the infinite power tower

$$\sqrt[n]{n}^{\sqrt[n]{n}^{\sqrt[n]{n}^{\cdot^{\cdot^{\cdot}}}}}$$

is a transcendental number. In particular

$$\sqrt[3]{3}^{\sqrt[3]{3}^{\sqrt[3]{3}^{\cdot^{\cdot^{\cdot}}}}}$$

is a transcendental number.

Finally, in the present paper an answer has been given to the Open Problem from [2]: To describe all rational numbers $a \in (1, e)$ and $b \in (e, +\infty)$ which are solutions of the equation:

$$\sqrt[a]{a} = \sqrt[b]{b}.$$

Namely, all rational solutions (of the above type) of the above equation are given by:

$$a = \left(1+\frac{1}{s}\right)^{s}, \quad b = \left(1+\frac{1}{s}\right)^{s+1}, \quad s = 1, 2, 3, \ldots$$

References

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